Non-Axisymmetric Equilibrium Reconstruction: V3FIT

James D. Hanson
Auburn University
(on sabbatical at Tech-X, Boulder, CO)
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Outline

• Explain: “Non-Axisymmetric Equilibrium Reconstruction – V3FIT”

• Example of reconstructions from CTH

• Probability and Inverse Problems – Posterior Covariance

• Magnetic diagnostics and eddy currents

• Summary and Conclusion
Non-Axisymmetric Equilibrium Reconstruction – V3FIT

- **Axisymmetric**
  - Idealized Tokamak

- **Non-Axisymmetric**
  - Stellarator / Torsatron
  - Real Tokamak
  - Reversed Field Pinch
Non-Axisymmetric **Equilibrium** Reconstruction – V3FIT

- **Ideal MagnetoHydroDynamics (MHD)**

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \\
\rho \frac{\partial \vec{v}}{\partial t} + \rho \vec{v} \cdot \nabla \vec{v} = \vec{J} \times \vec{B} - \nabla p \\
\left( \frac{\partial}{\partial t} + \vec{v} \cdot \nabla \right) \left( \rho \rho^{-\gamma} \right) = 0 \\
\vec{E} + \vec{v} \times \vec{B} = 0 \\
\nabla \cdot \vec{B} = 0 \\
\nabla \times \vec{B} = \mu_0 \vec{J}
\]

- **Equilibrium:** \( \frac{\partial}{\partial t} \rightarrow 0 \quad \vec{v} \rightarrow 0 \)

\[
\vec{J} \times \vec{B} = \nabla p \\
\nabla \cdot \vec{B} = 0 \\
\nabla \times \vec{B} = \mu_0 \vec{J}
\]

- **Ideal MHD Equilibrium** – answers the question: “Where is the plasma?”
Non-Axisymmetric Equilibrium Reconstruction – V3FIT

- Ideal MHD Equilibrium + Axisymmetry $\Rightarrow$ Grad-Shafranov Equation

$$B = \frac{1}{R} \nabla \psi \times \hat{e}_\phi + \frac{F(\psi)}{R} \hat{e}_\phi$$

$$\psi(R, Z) \quad p(\psi) \quad F(\psi)$$

$$\Delta^* \psi = -\mu_0 R^2 \frac{dp}{d\psi} - F \frac{dF}{d\psi}$$

- The magnetic field lies on surfaces of constant $\psi$, called flux surfaces.

- Need to specify the profile functions $p(\psi)$ and $F(\psi)$.

- The boundary conditions are vital.
  - Fixed-boundary – the outermost surface of the plasma is specified.
  - Free-boundary – the currents in external coils are specified.
Non-Axisymmetric Equilibrium Reconstruction – V3FIT

• Equilibrium reconstruction is the process of inferring the profiles $p$ and $F$ (and thus the full equilibrium) from measurements.

• Signals $S_i$
  – can be computed from the equilibrium (model signals)
  – can be measured from an experiment (observed signals)
  – Examples – magnetic diagnostic signals, soft x-ray signals

• Parameters $p_j$
  – Quantities that specify the equilibrium
    \[ p(\psi) = a_0 + a_1\psi + a_2\psi^2 + ... \]
  – Pressure and current profiles, external currents, total flux enclosed.

• Vary the parameters to minimize:
  \[ \chi^2 = \sum_i \left( \frac{S_i^{\text{model}}(p_j) - S_i^{\text{observe}}}{\sigma_i} \right)^2 \]

• Parameters at minimum of $\chi^2$ - these are equilibrium parameters.
Non-Axisymmetric Equilibrium Reconstruction – V3FIT

- Non-axisymmetry is *MUCH* harder than axisymmetry.
- Magnetic field-line flow is a Hamiltonian system (since $\nabla \cdot \vec{B} = 0$). Breaking axisymmetry corresponds to losing a conserved quantity (Noether’s theorem).
- Magnetic fields no longer need lie on flux surfaces.
  - Some field lines do lie on flux surfaces (black)
  - Some flux surfaces break up into chains of magnetic islands (blue)
  - Some field lines wander chaotically – fill a three-dimensional region of space (red)
Non-Axisymmetric Equilibrium Reconstruction – V3FIT

• Non-axisymmetric ideal MHD equilibrium – there is controversy about if this is a well-posed problem.
• Two main issues: the small-scale structure in the magnetic field, and the applicability of infinite conductivity to chaotic field lines.
• This has not stopped people from writing non-axisymmetric equilibrium codes.
• Assume good flux surfaces (no islands or chaotic regions):
• Do NOT assume good flux surfaces (allow magnetic islands and chaotic regions):
  – SIESTA [Hirshman, Sanchez, and Cook, *Physics of Plasmas* 18 062504 (2011)]
  – SPEC [Stuart Hudson, private communication]
Non-Axisymmetric Equilibrium Reconstruction – V3FIT

• **V3FIT is based on the VMEC 3-D equilibrium code**
  – VMEC is fast and robust – can get results between shots
  – VMEC can compute both free- and fixed-boundary equilibria
  – V3FIT iterations are closely coupled to the VMEC convergence
  – V3FIT has been benchmarked with the EFIT axisymmetric ER code
    
    `[Nuclear Fusion 49 075031 (2009)]`

• **V3FIT signal types (S):**
  – Magnetic diagnostics, Soft X-rays, Plasma limiters
  – Additional diagnostics under development

• **V3FIT Reconstruction Parameters (p):**
  – Pressure and current profile parameterizations
  – External currents
  – Total enclosed flux (size of the plasma)

• **V3FIT is in use on:**
  – CTH (Compact Toroidal Hybrid – Auburn University)
  – HSX (Helical Stellarator eXperiment – University of Wisconsin)
  – LHD (Large Helical Device –Toki, Japan – [Aaron Sontag – ORNL])
  – RFX (Reversed Field eXperiment – Padova, Italy – [David Terranova])
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CTH – Compact Toroidal Hybrid

- Auburn University, Steve Knowlton – Principal Investigator
- Torsatron with Ohmic heating coils
- Major radius 0.75 m
CTH – Compact Toroidal Hybrid
Reconstruction – Shot 11050236

- **Reconstruction Parameters (2)**
  - Phiedge \((\Phi_e)\) – total toroidal flux enclosed in plasma – size of plasma
  - Toroidal current profile parameter \(\alpha\):
    \[
    I(s) = \int \vec{J} \cdot \hat{e}_\phi \, d^2 A
    \]
    \[
    I'(s) = a_0 (1 - s^\alpha)^5
    \]
  
  Note: Radial coordinate \(s = \Phi / \Phi_e\) labels flux surfaces, \(0 \leq s \leq 1\)

- **Signals (27)**
  - Three 8-part Rogowski coils
  - One full Rogowski coil
  - Two plasma limiters
    - (at different toroidal angles).

- **One reconstruction takes about 3 minutes on a single processor.**
- **663 reconstructions**
CTH Shot 11050236

![Graph of plasma current over time](image)

- X-axis: Time (s)
- Y-axis: Plasma Current (MA)

The graph shows the plasma current over time, reaching a peak at approximately 1.64 seconds and decreasing rapidly thereafter.
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Probability and Inverse Problems

• **Forward problem** – start with a model, compute expected observations
  – Compute signals $S$ given model parameters $p$: $S_{\text{model}}(p)$
  – Drop a rock down a well. You know the depth of the well $h$. How long does it take to hit bottom $t_{\text{drop}}$?
  – For a VMEC-computed equilibrium, what signal do you expect in magnetic diagnostic # 27?

• **Equilibrium reconstruction is an Inverse Problem**
  – Infer model parameters from observed signals: $\rho(S_{\text{observe}})$
  – You heard the rock hit the water 2 seconds after you dropped it. How deep is the well?
  – Which VMEC parameters will best match all the magnetic diagnostic observations?

• **Probability** – quantification of uncertainty
  – Observed signals are not known to infinite accuracy.

• **Bayesian view of probability and statistics is most natural for physicists**

• **References I have found useful:**
Probability and Inverse Problems

\[ \chi^2 = \sum_i \left( \frac{S_{\text{model}}(p_j) - S_{\text{observe}}}{\sigma_i} \right)^2 \]

\[ = \left( \tilde{S}_{\text{model}}(\vec{p}) - \tilde{S}_{\text{observe}} \right)^T \cdot \underline{C}^{-1} \cdot \left( \tilde{S}_{\text{model}}(\vec{p}) - \tilde{S}_{\text{observe}} \right) \]

\[ \underline{C}^{-1} = \begin{pmatrix}
\sigma_1^{-2} & 0 & 0 \\
0 & \sigma_2^{-2} & 0 \\
0 & 0 & \ldots
\end{pmatrix} \]

\[ (\underline{C})_{ij} = \sigma_i \sigma_j \delta_{ij} \]

- C – Covariance matrix of the signal noise
- Specifying a probability distribution on space of signals.
- Assumption that signal noise is uncorrelated \( \Leftrightarrow \) C diagonal
- \( \underline{C}^{-1} \) acts like an inner product on the signal space - \( \chi \) is a dimensionless distance on the signal space.

\[ \langle \delta S_A | \delta S_B \rangle = \left( \delta \tilde{S}_A \right)^T \cdot \underline{C}^{-1} \cdot \left( \delta \tilde{S}_B \right) \]
Signal Space

- The model function $\mathcal{S}_{\text{model}}(\mathcal{P})$ takes a point in parameter space to a point in signal space.
- Usually, the signal space has much higher dimensionality that the parameter space.

- Solid curve – Image of a one-dimensional parameter space, $\mathcal{S}_{\text{model}}(\mathcal{P})$

- Red dot – observed signals $\mathcal{S}_{\text{observe}}$

- Dashed lines – probability distribution of observations
**Jacobian**

- The Jacobian relates small changes in parameters to small changes in signals:
  \[ (J)_{ij} = \frac{\partial S_i^\text{model} (\hat{p})}{\partial p_j} \quad \delta \hat{S} = J \cdot \delta \hat{p} \]

- The Jacobian can transfer the inner product \( C^{-1} \) on signal space into an inner product on parameter space.

\[
\langle \delta \tilde{p}_A | \delta \tilde{p}_B \rangle = \langle \delta \tilde{S}_A | \delta \tilde{S}_B \rangle \\
(\delta \tilde{p}_A)^T \cdot C_p^{-1} \cdot (\delta \tilde{p}_B) = (\delta \tilde{S}_A)^T \cdot C^{-1} \cdot (\delta \tilde{S}_B) \\
\Rightarrow \quad C_p^{-1} = J^T \cdot C^{-1} \cdot J \\
\]

\[
= (J \cdot \delta \tilde{p}_A)^T \cdot C_p^{-1} \cdot (J \cdot \delta \tilde{p}_B) \\
\]

- Interpret \( C_p \) as characterizing a probability distribution in parameter space.
Posterior Parameter Variance

- The diagonal elements of $C_p$ are the posterior parameter variances

\[ \sigma_{pj} = \sqrt{(C_p)_{jj}} \]

- $C_p$ need not be diagonal – the reconstructed parameters may be correlated.

- V3FIT computes the Jacobian in minimizing $\chi^2$. It also computes the posterior parameter variances.

- At right, black dots are reconstructed parameters, and gray lines indicate $\pm \sigma_{pj}$
Chi-Squared Distribution

• For Gaussian noise in the Signals, the minimum $\chi^2$ values are expected to have a chi-squared distribution.

• The CTH results do NOT have a chi-squared distribution.
Fly in the Ointment

• When plotted vs. time, the problem becomes apparent:

• The $\chi^2$ values are strongly correlated with the plasma current.

• Conclude:
  (1) There are systematic errors (not just Gaussian noise)
  (2) The model is not adequate

• Note: the model is both the equilibrium computation and the magnetic diagnostic signal computation.
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Magnetic Diagnostics

- Magnetic diagnostics measure the magnetic flux through a loop of wire.
  \[ \Phi_i = \int \mathbf{B} \cdot d\mathbf{A}_i = \oint \mathbf{A} \cdot d\mathbf{l}_i \quad \mathbf{B} = \nabla \times \mathbf{A} \]

- Outside the plasma, represent all current density as current in groups of filamentary wires. Then the flux integral can be simplified to
  \[ \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d\mathbf{r}' \]

- The model is now both the MHD equilibrium and the external current representation.
- For a fast and efficient way to compute the plasma contribution volume integral, see
CTH External Coils
CTH
Vaccum Vessel and Coil Frame
Magnetic Diagnostics and Eddy Currents

• Least accurately modeled current-carrying pieces of metal – the vacuum vessel and the coil frame.

• Measured currents in these structures are significant, induced by the ohmic loop voltage (and other time-changing magnetic fields)

• Auburn University requested assistance from Princeton Plasma Physics Laboratory “Offsite University Research Program” for help modeling the eddy currents in these complicated structures.

• Ali Zolfaghari and Dave Gates from PPPL.

• CAD models of structures -> eddy current modeling code SPARK.

• Translate results in format appropriate for VMEC input.
PPPL Results Translated to VMEC Format

Vacuum Vessel

Coil Frame

• Status – Checking results from PPPL
• Expect V3FIT reconstruction comparisons soon.
Summary

• **V3FIT Code Usage:**
  – CTH (Auburn University)
  – HSX (University of Wisconsin)
  – RFX (Padova, Italy)
  – LHD (Toki, Japan)

• **Magnetic Diagnostics and plasma limiter – well tested**

• **Soft x-rays tested, not yet in routine use**

• **Other diagnostics under development**

• **Applications of 3D equilibrium reconstruction**
  – Stellarators/ torsatrons
  – Tokamaks with significant non-axisymmetries
  – RFP’s in single-helicity state

• **Opinion –**
  – Equilibrium reconstruction imposes a well-structured discipline on both experimentalists and theorists/modelers.
  – Confronting discrepancies highlighted by equilibrium reconstruction helps improve both experiment and modeling.