Magnetohydrodynamics simulations of the reversed-field pinch with anisotropic thermal conductivity and nonuniform resistivity

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We present compressible magnetohydrodynamics (MHD) simulations of the reversed-field pinch in cylindrical coordinates, including density and pressure evolution.

- Isotropic thermal conductivity

- Anisotropic thermal conductivity

- Nonuniform resistivity with a given radial profile

- Resistivity is a function of temperature $\eta = \eta_0 T^{-3/2}$
Compressible MHD equations

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0
\]

\[
\frac{\partial \rho \mathbf{v}}{\partial t} = -\nabla \cdot \left( \rho \mathbf{v} \mathbf{v} + P \delta_{ij} - \nu \sigma_{ij} \right) + \mathbf{J} \times \mathbf{B}
\]

\[
\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \left( \eta \mathbf{J} - \mathbf{v} \times \mathbf{B} \right)
\]

\[
\frac{\partial P}{\partial t} = -\left[ \nabla \cdot (\mathbf{v} P - \kappa \nabla T) + (\gamma - 1) P \nabla \cdot \mathbf{v} \right] + (\gamma - 1) H_p
\]

\[
H_p = \eta J^2 + \nu \left[ \frac{1}{2} \sigma_{ij} \sigma_{ij} - \frac{2}{3} \nabla \cdot \mathbf{v} \right]
\]

Cylindrical coordinates

\[
0 < r < 1 \quad 0 < \theta < 2\pi \quad z = R\phi \quad (0 < \phi < 2\pi)
\]
Boundary conditions

Periodic boundary conditions in \( \vartheta \) and \( z \)

In the radial direction we assume a conducting wall in the contact with the plasma

\[
B_r = 0 \quad V = 0 \quad J_{\vartheta}^{(m,n)} = 0 \quad J_z^{(m,n)} = 0
\]

\[
B_{\vartheta}^{(0,0)} = \text{const} \quad B_z^{(0,0)} = \text{const} \quad T = \text{const}
\]

The equations are solved with a pseudospectral method in the periodic direction, finite differences in the radial direction and an explicit Runge-Kutta time scheme
We find that most of the energy is contained in the diagonal line \( n/m = -2R \) in the \((m,n)\) plane. In order to obtain a good resolution with a low number of Fourier harmonics we perform a change of coordinates

\[
\phi = \vartheta - 2z \\
z' = z
\]

so that the new mode numbers

\[
m' = m \\
n' = 2mR + n
\]
Simulation with isotropic thermal conductivity

\[ \eta = 10^{-3}, \quad \nu = 10^{-3}, \quad \kappa = 10^{-3} \quad R = 4 \]

A multiple helicity state is formed

A hot island is present, but the magnetic field is chaotic
The thermal conductivity in a magnetized plasma is anisotropic with respect to the direction of the magnetic field and for a fusion plasma the ratio $\frac{\kappa_||}{\kappa_\perp}$ may exceed $10^{10}$.

Thermal conduction occurs on different time scales in the parallel and perpendicular direction, so that magnetic field lines tend to become isothermal.

In a simulation it is not possible to use a realistic value of $\kappa_||$ because the time step would become too small.
Multiple-time-scale analysis
(Davidson, 1972, Onofri et al. PoP 2011)

We separate the evolution on fast time scales from the evolution on slow time scales

\[ \frac{dP}{dt} = -\varepsilon \left[ \nabla \cdot (VP - \kappa_\perp \nabla_\perp T) + (\gamma - 1)P \nabla \cdot V + (\gamma - 1)H_p \right] + \nabla \cdot (\kappa_\parallel \nabla_\parallel T) \]  

(\varepsilon << 1)

Expand \( P \) in the small parameter \( \varepsilon \)

\[ P = P_0 + \varepsilon P_1 + \varepsilon^2 P_2 + ... \]

Extend the number of time variables

\[ \frac{d\tau_0}{dt} = 1, \quad \frac{d\tau_1}{dt} = \varepsilon, \quad \frac{d\tau_2}{dt} = \varepsilon^2, \quad \ldots \]

We obtain an equation for each order in \( \varepsilon \)
Extending the number of time variables introduces new solutions which are not present in the original equations. Such solutions may contain time secularities, which would invalidate the expansion

\[ P = P_0 + \varepsilon P_1 + \varepsilon^2 P_2 + \ldots \]

The freedom introduced by extending the time variables can be used to remove, order by order, the time secularities in the solution.

The condition that the first-order solution \( P_1 \) must be nonsecular as \( \tau_0 \to \infty \) determines the evolution of the zeroth-order solution \( P_0 \) on the \( \tau_1 \) time scale

\[
\frac{\partial P_0}{\partial \tau_0} = \nabla \cdot \left[ \kappa_\parallel \nabla_\parallel T \right] \quad (1)
\]

\[
\frac{\partial P_0}{\partial \tau_1} = -\left[ \nabla \cdot (\nabla P_0 - \kappa_\perp \nabla_\perp T) + (\gamma - 1) P_0 \nabla \cdot V \right] + (\gamma - 1) H_p \quad (2)
\]

At each time step we look for an asymptotic solution of (1) and use it in (2)
The simulations produce hot structures corresponding to closed magnetic surfaces in SH states and almost flat temperature when the magnetic field is chaotic in MH states.
Radial profile of temperature in MH and SH

Poincare section of the dominant mode magnetic field in the SH state

A Magnetic island and an X-point are present
In RFP experiments, the resistivity increases near the wall, where temperature is lower.

\[ \eta(r) = \eta_0 \left(1 + 19r^{10}\right) \]

\[ \eta_0 = 10^{-4} \]
Nonuniform resistivity (SH)

\[ \eta(r) = 10^{-4}(1 + 19r^{10}) \]

Uniform resistivity (MH)

\[ \eta = 10^{-4} \]

The simulation with uniform resistivity show the formation of MH states, while a stationary SH state is found when the resistivity is radially increasing, with the same on-axis value.
Reversal parameter \( F = \frac{B_z}{\langle B_z \rangle} \) and pinch parameter \( \Theta = \frac{B_{\phi}}{\langle B_z \rangle} \)

\[ \text{Uniform resistivity} \]

\[ \text{Nonuniform resistivity} \]

\begin{align*}
F & \quad \Theta \\
\downarrow & \quad \downarrow \\
-0.050 & \quad -0.030 & \quad -0.028 \\
-0.045 & \quad -0.035 & \quad -0.032 \\
-0.040 & \quad -0.030 & \quad -0.026 \\
-0.035 & \quad -0.025 & \quad -0.024 \\
-0.040 & \quad -0.030 & \quad -0.026 \\
-0.045 & \quad -0.035 & \quad -0.028 \\
-0.050 & \quad -0.040 & \quad -0.032 \\
\end{align*}

\begin{align*}
0 & \quad 1000 & \quad 2000 & \quad 3000 & \quad 4000 \\
\end{align*}
Poincaré section of magnetic field lines of the dominant mode in the poloidal plane $z = 0$

The SH state found in the simulation with nonuniform resistivity is a single-helical axis state (SHAx), the separatrix of the magnetic island disappears and a single helical magnetic axis exists.
\[ \eta = \eta_0 T^{-\frac{3}{2}} \]
Including density and pressure evolution modifies the results.

With anisotropic thermal conductivity, MHD simulations produce almost flat temperature profiles in MH states and a hot island in SH states.

The resistivity profile is important in the formation of SH states in the RFP. Two simulations with the same viscosity and on-axis resistivity show the formation of SH and MH states, depending on the radial resistivity profile.

The amplitude of the dominant mode increases with the ratio between the temperature in the core and at the wall.