Resistive Magnetohydrodynamic Equilibrium and Stability of a Rotating Plasma with Particle Sources

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Basic Ideas of Centrifugal Confinement

- Use centrifugal forces from rotation to confine the plasma along field lines.

\[ \mathbf{B} \cdot \nabla p = -\mathbf{B} \cdot (\rho \mathbf{u} \cdot \nabla \mathbf{u}) \sim 1 : M_S^2 \]

- Velocity shear could stabilize interchanges.
Simple Mirrors are Unstable to Flutes

\[ g_{\text{eff}} \sim \frac{C_s^2}{R_c} \]

\[ \gamma g \sim \sqrt{\frac{C_s'^2}{R_C L_p}} \]
Flutes in Simple Mirror - Simulation

- No rotation:
  \[ B \cdot \nabla p = 0 \]

- \[ V_A \sim C_S = 1 \]
Centrifugal Force can also Drive Flutes

This is analogous to the Rayleigh-Taylor Instability that occurs when a heavy fluid sits on top of a light fluid.

\[ \gamma_g \sim \sqrt{\frac{g}{L_\rho}} \]

\[ g \sim \frac{u_\phi^2}{r} \]
$V'$ Stabilization of Flutes

$V'$ tears apart convection cells of interchanges ⇒ shorten the length scales ⇒ non-ideal effects become important.

$$V' > \gamma_g (\ln(R_{\mu}))^{1/2} \Rightarrow \text{Stability}$$

(Hassam, Phys. Fluids. B. 1992)

- Non-ideal effects are essential for $V'$ stabilization. Perturbations grow algebraically in the ideal limit.
- Finite Lamor radii effect is stabilizing. See the poster of Sheung-Wah Ng.
Kelvin-Helmholtz Instability

Velocity shear is also a potential instability driver...

Photo by Paul E. Branstine, Weatherwise magazine.

M. van Dyke *An album of fluid motion*
NMCX – 2D Laminar Profiles

Temperature

Pressure

Temperature

Pressure
NMCX – Velocity Shear Stabilization

Huang & Hassam, PRL (87), 2001
Motivation

The 2-D steady state of NMCX was in fact, not steady; it was still slowly evolving on transport time scales. If the resistivity is taken into account:

- Straight field and no particle sources $\Rightarrow J_\phi \rightarrow 0$ due to resistivity $\Rightarrow$ density piles up towards the outside.
- Curved field $\Rightarrow$ poloidal convections $\Rightarrow J_\phi \neq 0$ $\Rightarrow$

What is the steady state in the absence of particle sources?

NMCX is very stable, no wobbles as appeared in a previous Z-pinch simulation.

Is this a general result?
Resistive MHD with Particle Sources

Continuity equation with particle sources:

\[ \nabla \cdot (\rho u) = S, \]

Momentum equation:

\[ \rho u \cdot \nabla u = -\nabla p - \frac{\nabla B^2}{2} + B \cdot \nabla B, \]

Ohm’s law in the simplest form:

\[ E = -\nabla \Phi = -u \times B + \eta J, \]

Ampere’s law:

\[ J = \nabla \times B. \]
Particle Sources, Con’t

Assume purely poloidal field $B = \nabla \phi \times \nabla \psi$,

$$ J = \nabla \times B = r^2 \nabla \cdot \left( \frac{\nabla \psi}{r^2} \right) \nabla \phi $$

Take momentum equation along $\nabla \psi \Rightarrow$

$$ (\rho r \Omega^2 \hat{r} - \nabla p) \cdot \nabla \psi = \nabla \cdot \left( \frac{\nabla \psi}{r^2} \right) |\nabla \psi|^2 $$

Ohm’s law along $\hat{\phi} \Rightarrow$

$$ u \cdot \nabla \psi = \eta r^2 \nabla \cdot \left( \frac{\nabla \psi}{r^2} \right) $$

$$ = \eta r^2 \frac{(\rho r \Omega^2 \hat{r} - \nabla p) \cdot \nabla \psi}{|\nabla \psi|^2} $$
Integrate $\nabla \cdot (\rho u) = S$:

$$
\int_{\psi \leq c} S d\tau = \int_{\psi = c} \rho u \cdot \nabla \psi \frac{d\sigma}{|\nabla \psi|}
$$

$$
= \eta \int_{\psi = c} \rho r^2 (\rho r \Omega^2 \hat{r} - \nabla p) \cdot \nabla \psi \frac{d\sigma}{|\nabla \psi|^3}
$$

$$
= \eta \int_{\psi = c} \rho (\rho r \Omega^2 \hat{r} - \nabla p) \cdot \hat{n} d\sigma \frac{B^2}{|\nabla \psi|}
$$
Particle Sources, Corollary

$$\int_{\psi \leq c} S d\tau = \eta \int_{\psi = c} \frac{\rho (\rho r \Omega^2 \hat{r} - \nabla p) \cdot \hat{n} d\sigma}{B^2}$$

- $S = 0 \Rightarrow \rho r \Omega^2 \hat{r} \sim \nabla p$
  $\Rightarrow$ **Density piles up exponentially**

- $\Omega = 0 \Rightarrow S \sim \eta \beta \rho / a^2$.  
  $\Omega \neq 0 \Rightarrow S \sim (1 + M^2_S(a/r)) \eta \beta \rho / a^2$
  $\Rightarrow$ **Centrifugal force enhances cross-field particle loss**

- Density profile could be controlled by particle sources.
  $\Rightarrow$ **Additional knob for stability**
Dean Flow Model

Hassam, Phys. Plasmas, 1999
Dean Flow – Analytic Results

- $V'$ stabilization:

$$r^2 \Omega'^2 > ( -r \Omega^2 \rho' / \rho - gp'/p ) \ln(R_\mu)$$

$$\sim 1 : ( a/r : a / R_c M_S^2 ) \ln(R_\mu)$$

- High $M_S$ can stabilize $p'$ driven interchanges.
- $\rho'$ driven interchanges are harder to stabilize.

- Generalized Rayleigh’s Inflexion Theorem

$$\frac{d}{dr} \left( \frac{\rho}{r} \frac{d(r^2 \Omega)}{dr} \right) \neq 0 \Rightarrow \text{KH ideally stable}$$

- Density profile is also crucial to KH.
Numerical Model

\[ \partial_t \rho + \nabla \cdot (\rho u) = S, \]

\[ \rho \partial_t u + \rho u \cdot \nabla u = -T \nabla \rho - \frac{\nabla B^2}{2} + B \cdot \nabla B + \mu \nabla^2 u + F \hat{\phi}, \]

\[ \partial_t B = -\nabla \times E, \]

\[ E = -u \times B + \eta \nabla \times B. \]

- \[ \partial_z = 0, \text{ 2-D.} \]
- \[ T = const, \text{ no effective gravity.} \]
- Flow is driven by \( F \).
- \( S \) is calculated to dial-in desired density profiles.
\[ \rho \partial_t \mathbf{u} + \rho \mathbf{u} \cdot \nabla \mathbf{u} = -T \nabla \rho - \frac{\nabla B^2}{2} + \mathbf{B} \cdot \nabla \mathbf{B} + \mu \nabla^2 \mathbf{u} + \rho r \Omega_0^2 \hat{r} \]
Stabilization: Aspect Ratio

The system approaches laminar as $R$ increases.

$\nabla'$

$R = 4$

$R = 2$

$R = 4$

$R = 6$

$R = 8$

$R = 10$
Deviation from the laminar state gets larger as $\mu$ or $\eta$ become smaller.
Interchanges are localized about where $\Omega' = 0$. If $\rho'$ is stabilizing there, the system could be completely stable.
Kelvin-Helmholtz Instability

K-H modes could be unstable when generalized Rayleigh’s criterion is not satisfied.
K-H and Interchanges

K-H modes and interchanges both occur when generalized Rayleigh’s criterion is not satisfied and $\rho'$ is destabilizing at the weakest point.
Why NMCX so Stable?

Both \( \rho' \) and \( \rho' \) are stabilizing at the weakest point.
Summary

- Particle sources are necessary for a steady state of centrifugally confined plasmas. The density profile of the steady state depends on the placement of particle sources.

- The slowly diffusing steady state is realizable only when it is stable.
  - $V'$ stabilization: localized interchanges about where $\Omega' = 0$; “transport barrier” is largely maintained.
  - K-H modes could occur when generalized Rayleigh’s criterion is not satisfied.

- With judicious placement of particle sources, a completely stable steady state could be achieved. Particle sources could be utilized to optimize the stability.