PHYSICALLY MOTIVATED AND MATHEMATICALLY CONSISTENT MAGNETIC HELICITY

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MAGNETIC HELICITY

Magnetic helicity $K$ is a measure of the flux linkage of a magnetic field $\mathbf{B}$.

Moffatt’s example of magnetic helicity…linked flux loops:

$$
K \equiv 2 \Phi_1 \Phi_2 = \int_{V=V_1+V_2} \mathbf{A} \cdot \mathbf{B} \, d^3x,
\quad d^3x = \frac{\Phi}{B} \, d\ell
$$

Linkage of a vector field is a global topological property.

In general must integrate over all space.

Helicity is a property of a complete object (magnetic field). But what if $\mathbf{B}$ is known and/or of interest in only a limited region of space which links and connects with outside fluxes?

Local or differential magnetic helicity has no logically justified physical interpretation.
RELATIVE Helicity Has Physical Meaning

Following Berger & Field, \textit{J. Fluid Mech.} \textbf{147} (1984) 133, \textbf{Relative Helicity} $K_{\text{rel}}$ is the helicity \textit{difference} between two fields in \textbf{all space}, physical field $B$ and a chosen reference field $B_{\text{ref}}$ outside the volume $V$ of interest.

Let $B_{\text{ref}} = B$ outside of the volume $V$ of interest, i.e., throughout $V_{\beta}$.

Because the linkages contributed by $B_{\text{ref}}$ and $B$ outside $V$ are equal, they do not contribute to the \textbf{RELATIVE} helicity.

\[
K_{\text{rel}} = \int_{V_{\infty}} A \cdot B d^3x - \int_{V_{\infty}} A_{\text{ref}} \cdot B_{\text{ref}} d^3x
\]

$B = \nabla \times A$, \quad $B_{\text{ref}} = \nabla \times A_{\text{ref}}$
How to Define and Use Relative Helicity Has Not Been Made Clear

- Berger-Field did not extend their ideas to toroidally connected volumes, and some of their proofs were restricted to the Coulomb gauge, $\nabla \cdot \mathbf{A} = 0$, and to $\nabla \times \mathbf{B} = 0$.
  - Many authors follow Bevir-Gray and add a simple external flux linkage term, but then the toroidal surface must be a magnetic surface.
  - Extension of Berger-Field requires either restriction on gauge of $\mathbf{A}$ or extra terms, e.g., Finn & Antonsen.
  - Alternatively, the torus is cut, yielding a simply connected volume. Must choose between magnetic surface or magnetically penetrated surface boundary cases.
- Boozer treated moving and deforming, toroidally and simply connected boundaries, but not with a relative helicity.
  - Moving boundary can change (a) linkage within $V$ and (b) connection to outside.
  - Limited to stationary, simply connected volumes, a special gauge, and does not explicitly discuss relative helicity; but might still be valid.
- Need to formulate a physically meaningful, mathematically rigorous magnetic helicity and rules for its use, for at least
  
  toroidally and/or simply connected,
  arbitrarily penetrated,
  moving boundary
TOWARD A GENERALIZED MAGNETIC HELICITY
(a work in progress)

Begin with BF relative helicity over all space, \( V_\infty = V_\alpha + V_\beta \), with \( B_{\text{ref}} = B \) throughout \( V_\beta \):

\[
K_{\text{rel}} = \int_{V_\infty} A \cdot B \, d^3x - \int_{V_\infty} A_{\text{ref}} \cdot B_{\text{ref}} \, d^3x
\]

\[
= \int_{V_\alpha = V} \left( A \cdot B - A_{\text{ref}} \cdot B_{\text{ref}} \right) \, d^3x + \int_{V_\beta} \left( A - A_{\text{ref}} \right) \cdot B \, d^3x
\]

Since \( B_{\text{ref}} = B \) in \( V_\beta \), let \( A - A_{\text{ref}} = \nabla f_\beta \) in \( V_\beta \), with \( f_\beta \) globally well-defined, single-valued in \( V_\beta \).

Require that all sources be within \( V_\infty \), so \( B \) does not contribute to integrals at \( \infty \).

Let \( S \) be the (closed) surface that encloses \( V \) and \( n \) its outward unit normal vector.

Then,

\[
K_{\text{rel}} = \int_V \left( A \cdot B - A_{\text{ref}} \cdot B_{\text{ref}} \right) \, d^3x - \int_S f_\beta B \cdot n \, d^2x
\]

The surface integral is commonly overlooked or else eliminated by restrictive choice(s).

Here it is kept.
TOWARD A GENERALIZED MAGNETIC HELICITY (2)

Add some physics:

B (real field) and B_{ref} (reference field) should both be physically realizable in V_{∞} to have meaning.

Then, the following field components must each be continuous across S:

\[ \mathbf{B} \cdot \mathbf{n}, \quad \mathbf{B}_{\text{ref}} \cdot \mathbf{n}, \quad \mathbf{A} \times \mathbf{n}, \quad \mathbf{A}_{\text{ref}} \times \mathbf{n}, \quad \nabla g \times \mathbf{n}, \quad \nabla g_{\text{ref}} \times \mathbf{n}, \quad g, \quad g_{\text{ref}} \]

\nabla g is any globally well-defined gauge function. Continuity of g means no charge double layer.
Continuity of normal B means no magnetic charge on S.
Continuity of tangential A and \nabla g means no magnetic flux sheets in S.

Therefore, \nabla f_{\alpha} \times \mathbf{n} = \nabla f_{\beta} \times \mathbf{n} across S. Because of continuity, can drop subscripts \(\alpha, \beta\) on S.

Then, the generalized relative helicity is

\[
K_{\text{rel}} = \int_{V} \left( \mathbf{A} \cdot \mathbf{B} - \mathbf{A}_{\text{ref}} \cdot \mathbf{B}_{\text{ref}} \right) d^{3}x - \int_{S} f \mathbf{B} \cdot \mathbf{n} d^{2}x
\]

where \(f\) must satisfy \nabla f \times \mathbf{n} = (\mathbf{A} - \mathbf{A}_{\text{ref}}) \times \mathbf{n} on S.

\(K_{\text{rel}}\) is independent of gauges \nabla g and \nabla g_{\text{ref}} added to \(\mathbf{A}\) and \(\mathbf{A}_{\text{ref}}\) and depends only on quantities in \(V\) and on \(S\), i.e., the “region of interest”.

(but remember to add \(g - g_{\text{ref}} + \text{(any constant)}\) to \(f\).
Decompose B into “CLOSED” and “OPEN” Components

Definitions:

Let \( B = B_{\text{cl}} + B_{\text{op}} \) and \( B_{\text{ref}} = B_{\text{ref,cl}} + B_{\text{ref,op}} \) \( \text{with } \nabla \cdot B_{\text{cl}} = \nabla \cdot B_{\text{op}} = \nabla \cdot B_{\text{ref,cl}} = \nabla \cdot B_{\text{ref,op}} = 0. \)

Require **closed** components to satisfy \( B_{\text{cl}} \cdot n = B_{\text{ref,cl}} \cdot n = 0 \) on S

Require **open** components to satisfy \( B_{\text{op}} \cdot n = B_{\text{ref,op}} \cdot n \neq 0 \) on S

Also let there be corresponding vector potentials that satisfy

\[
\begin{align*}
B_{\text{cl}} &= \nabla \times A_{\text{cl}} \\
B_{\text{op}} &= \nabla \times A_{\text{op}} \\
B_{\text{ref,cl}} &= \nabla \times A_{\text{ref,cl}} \\
B_{\text{ref,op}} &= \nabla \times A_{\text{ref,op}}
\end{align*}
\]

Closed B is fully contained within V and does not penetrate S; S is a magnetic surface of \( B_{\text{cl}} \).

Open B penetrates S and connects with the outside.

This decomposition is not unique.

Relative helicity then becomes

\[
K_{\text{rel}} = \int_V \left( A_{\text{cl}} \cdot B_{\text{cl}} + A_{\text{op}} \cdot B_{\text{op}} + A_{\text{cl}} \cdot B_{\text{op}} + A_{\text{op}} \cdot B_{\text{cl}} \right) d^3x
\]

\[
-\int_V \left( A_{\text{ref,cl}} \cdot B_{\text{ref,cl}} + A_{\text{ref,op}} \cdot B_{\text{ref,op}} + A_{\text{ref,cl}} \cdot B_{\text{ref,op}} + A_{\text{ref,op}} \cdot B_{\text{ref,cl}} \right) d^3x
\]

\[
-\int_S \left( f_{\text{cl}} \, B_{\text{cl}} + f_{\text{op}} \, B_{\text{op}} + f_{\text{cl}} \, B_{\text{op}} + f_{\text{op}} \, B_{\text{cl}} \right) \cdot n \, d^2x
\]
...and Simplify It...

The preceding $K_{\text{rel}}$ expression simplifies greatly, yet still has physical content, if we require that

$$B_{\text{op}} = B_{\text{ref,op}}$$

everywhere in $V$.

This choice also makes the closed-open decomposition unique.

Many terms cancel, and $K_{\text{rel}}$ simplifies to

$$K_{\text{rel}} = \int_V \left[ A_{\text{cl}} \cdot B_{\text{cl}} - A_{\text{ref,cl}} \cdot B_{\text{ref,cl}} + 2 \left( B_{\text{cl}} - B_{\text{ref,cl}} \right) \cdot A_{\text{op}} \right] d^3x$$

The last integrand term can also be made equal to

$$+ 2 \left( A_{\text{cl}} - A_{\text{ref,cl}} \right) \cdot B_{\text{op}}$$

(using $B_{\text{cl}} \cdot n = 0$ and $(A_{\text{cl}} - A_{\text{ref,cl}}) \times n = \nabla f \times n$ on $S$)

Interpretation:

Relative helicity in toroidally and simply connected volumes can be reduced to the relative helicity of just a closed field component, plus (if $S$ is penetrated by $B$) a cross linkage between closed and open field components.
So far, we have neither restricted gauge choice nor required that $B_{\text{ref}}$ or its open or closed components be source-free (current-free).

Now consider special case where:

- $B_{\text{op}}$ is current-free, a common choice
  - “Vacuum” $B$ is the lowest energy field in $V$ with given boundary conditions
  - Then let $B_{\text{op}} = B_{\text{ref,op}} = \nabla \chi$ in $V$, where $\chi$ is a well-defined scalar potential in $V$
- $A_{\text{op}}$ and $A_{\text{ref,op}}$ gauges chosen so that each has the same “electric charge”
  - Then $\nabla \cdot (A_{\text{op}} - A_{\text{ref,op}}) = 0$ in $V$ and $(A_{\text{op}} - A_{\text{ref,op}}) \cdot n = 0$ on $S$.

This is a less restrictive version of the Moses et al. gauge choice.

Then, the third term of $K_{\text{rel}}$ integrates to zero, and

$$K_{\text{rel}} = \int_V \left( A_{\text{cl}} \cdot B_{\text{cl}} - A_{\text{ref,cl}} \cdot B_{\text{ref,cl}} \right) d^3x$$

This looks like a “closed field only” version of the BF relative helicity equation.
Differentiate $K_{\text{rel}}$ with respect to time:

$$\frac{dK_{\text{rel}}}{dt} = \frac{d}{dt} \int \mathbf{A} \cdot \mathbf{B} \, d^3x - \frac{d}{dt} \int \mathbf{A}_{\text{ref}} \cdot \mathbf{B}_{\text{ref}} \, d^3x - \frac{d}{dt} \int f \mathbf{B} \cdot \mathbf{n} \, d^3x$$

$$= \int \frac{\partial}{\partial t} \left( \mathbf{A} \cdot \mathbf{B} \right) \, d^3x + \int \left( \mathbf{A} \cdot \mathbf{B} \right) \left( \mathbf{U} \cdot \mathbf{n} \right) \, d^2x$$

$$- \int \frac{\partial}{\partial t} \left( \mathbf{A}_{\text{ref}} \cdot \mathbf{B}_{\text{ref}} \right) \, d^3x + \int \left( \mathbf{A}_{\text{ref}} \cdot \mathbf{B}_{\text{ref}} \right) \left( \mathbf{U} \cdot \mathbf{n} \right) \, d^2x$$

$$- \frac{d}{dt} \int f \mathbf{B} \cdot \mathbf{n} \, d^3x$$

In what follows, I will use $\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla \phi$, where $\phi$ is a scalar electric potential.

This helps to distinguish electrostatic from inductive effects.
We need EM fields in the moving frame of S(x,t), to apply boundary conditions at S.

Let \( \mathbf{x} \) be the coordinate vector in a “fixed” frame. Let \( \mathbf{U}(\mathbf{x},t) \) be the local normal component of the velocity of S(t) at \( \mathbf{x} \). \( \mathbf{U} \times \mathbf{n} = 0 \). In particular, let \( \mathbf{U} \) be the stream line field of a set of coordinate system points attached to S(x,t). Let prime (’) denote a value measured at a point \( \mathbf{x} \) in an inertial frame moving with the local \( \mathbf{U}(\mathbf{x},t) \).

Let \( \frac{\partial \mathbf{A}'}{\partial t} \) denote the temporal variation of \( \mathbf{A}' \) measured by an observer moving with a unique coordinate point on \( S(\mathbf{x},t) \), while \( \frac{d\mathbf{A}}{dt} \) is the convective derivative of \( \mathbf{A} \) along \( \mathbf{U} \) in the fixed frame.

Non-relativistic transformations between fixed and moving frames are:

\[
\begin{align*}
B' &= B \\
A' &= A \\
E' &= E + \mathbf{U} \times B \\
\phi' &= \phi - \mathbf{U} \cdot \mathbf{A} \\
\frac{\partial \mathbf{A}'}{\partial t} &= \frac{\partial \mathbf{A}}{\partial t} - \mathbf{U} \times B + \nabla(\mathbf{U} \cdot \mathbf{A}) = \frac{\partial \mathbf{A}}{\partial t} + \mathbf{U} \cdot \nabla \mathbf{A} = \frac{d\mathbf{A}}{dt}
\end{align*}
\]

Tangential components of \( \mathbf{E} \) are continuous across \( S \) in the moving frame.
Upon expanding the $\frac{d}{dt}$ of the surface integral, many terms cancel, leaving

$$\frac{dK_{\text{rel}}}{dt} = \int_{V(t)} \frac{\partial}{\partial t} \left( \mathbf{A} \cdot \mathbf{B} - \mathbf{A}_{\text{ref}} \cdot \mathbf{B}_{\text{ref}} \right) d^3x - \int_{S(t)} \frac{\partial (fB)}{\partial t} \cdot \mathbf{n} d^3x$$

- Only partial time derivatives appear in this form

Further algebra and the boundary condition $(\mathbf{E}' - \mathbf{E}'_{\text{ref}}) \times \mathbf{n} = 0$ on a moving surface $S(t)$ yields

$$\frac{dK_{\text{rel}}}{dt} = -\int_{V(t)} 2 \mathbf{E} \cdot \mathbf{B} d^3x + \int_{V(t)} 2 \mathbf{E}_{\text{ref}} \cdot \mathbf{B}_{\text{ref}} d^3x$$

- This has the same simple form as some prior developments of helicity evolution
- Further physics is needed to evaluate the $\mathbf{E} \cdot \mathbf{B}$ integrals
Plasma physics, including helicity and current source and sink terms, are added in a constitutive relation, a.k.a. Ohm’s law (A. Boozer, this conference).

The Braginskii two-fluid plasma Ohm’s law yields

\[ \mathbf{E} \cdot \mathbf{B} = \eta_{||} \mathbf{J} \cdot \mathbf{B} - \frac{1}{en_e} \nabla n_e \cdot \mathbf{B} - \frac{0.7k}{e} \nabla T_e \cdot \mathbf{B} \]

- Parallel gradients of \( T_e \) and \( n_e \) or \( p_e \) can contribute to or subtract from helicity dissipation
  - Especially when B-lines are so long that particle density deviates from a line invariant
  - Hall \( J_e \times B \) term plays no role
SUMMARY

- Substantial progress has been made in an effort to formulate a broadly applicable, physically motivated, mathematically consistent relative magnetic helicity based on Berger-Field principles
  - Toroidally and simply connected volumes on an equal basis
  - No restrictions on B-field penetrations of bounding surface
  - Continuity and matching conditions are physically motivated, not *ad hoc*
  - Much progress can be made with no restrictive gauge requirement.
  - Decomposition of the magnetic field into closed and open components lends further insight
    - Relative helicity can be defined in terms of closed field components alone
- Helicity evolution equation has been partly formulated
  - Moving and deforming boundary with unrestricted B-field penetrations
  - No restrictive requirements imposed so far
- **Remaining Work:**
  - Evaluate and interpret $E_{\text{ref}} \cdot B_{\text{ref}}$ integral.
  - Decompose B into closed and open components
“Appendix on Measurement”
Helicity transported across a closed surface $S(x,t)$ bounding a volume $V(x,t)$ can be written as

$$\text{Helicity out of } S = \int_{S(t)} \left( A' \times \frac{\partial A'}{\partial t} + \phi' B' \right) \cdot n \, d^2x$$

With Moses’ gauge for $A$, we can drop the primes (moving coordinate system on $S$) and use variables in fixed lab frame; vector potential term drops out, so

$$\int_{S(t)} A' \times \frac{\partial A'}{\partial t} \cdot n \, d^2x = 0.$$ 

Therefore, we would only need to measure the simple helicity flux term $\overline{\phi B} \cdot n$

...just $B(t)$ normal to the plasma average magnetic surface and the true plasma potential

UCSD harmonic probe measures plasma potential.
Simple Helicity Flux Probe Concept:
Measure Local Plasma Potential and Normal Component of $B$