MHD equilibrium and stability of rotating plasmas in a mirror geometry

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Geometry and Equations

- A straight cylinder, with a perfectly conducting wall and periodically identified ends.

- Azimuthal (poloidal) symmetry.

- \( z = R\zeta \), with \( R = 1 \).

- Minor radius \( a = 1 \).

We use the \textit{single fluid} ideal MHD equations, with some dissipative terms added for numerical stability.
Single fluid resistive MHD equations

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \]

\[ \rho \frac{\partial \mathbf{u}}{\partial t} + \rho (\mathbf{u} \cdot \nabla) \mathbf{u} = \mathbf{J} \times \mathbf{B} - \nabla p, \]

\[ \frac{\partial \mathbf{A}}{\partial t} = \mathbf{u} \times \mathbf{B} - \eta \mathbf{J}, \]

\[ \frac{\partial p}{\partial t} + (\mathbf{u} \cdot \nabla) p + \gamma p \nabla \cdot \mathbf{u} = \kappa_\perp \nabla^2 p + \kappa_\parallel \nabla_\parallel^2 (p/\rho), \]

\[ \mathbf{B} = \nabla \times \mathbf{A}, \quad \mathbf{J} = \nabla \times \mathbf{B}. \]
General observations on equilibrium

• Properties of MHD equilibria with flows have been examined by many authors.

The $\mathbf{B}$-field:

• With $\partial/\partial \theta = 0$, the equilibrium mirror field can be derived from $A_\theta$ only.

• Thus, we let $\mathbf{B} = \nabla \times r \nabla \theta A_\theta = \nabla \chi \times \nabla \theta$, where $\chi \equiv r A_\theta$. 
General observations on equilibrium:

force balance

- Momentum equation: \( \rho (\mathbf{u} \cdot \nabla) \mathbf{u} = \mathbf{J} \times \mathbf{B} - \nabla p \).
- Using \( \mathbf{u} = r^2 \Omega(\chi, s) \nabla \theta \), and \( \mathbf{B} = \nabla \chi \times \nabla \theta \), we get:
- \( (1/2) \rho \Omega^2 \nabla r^2 = (1/r^2) \nabla^2 \chi \nabla \chi + \nabla p \), with the following two components:
- \( (1/r^2) \nabla^2 \chi = (1/2) \rho \Omega^2 \partial r^2 / \partial \chi - \partial p / \partial \chi \), “Grad-Shafranov equation”, and
- \( \partial p / \partial s = (1/2) \rho \Omega^2 \partial r^2 / \partial s \), which represents force balance within a flux surface. Note that in general, \( r = r(\chi, s) \), and \( p \) is not a flux function.
General observations on equilibrium: 

\textit{parallel force balance}

- Parallel force balance: \( \frac{\partial p}{\partial s} = (1/2)\rho \Omega^2 \frac{\partial r^2}{\partial s} \).

- Assume \( T = T(\chi) \), fast parallel thermal conduction, and define \( c_s \equiv \sqrt{2kT(\chi)/m} \), the sound speed, to obtain:

\( (\partial/\partial s) \ln \rho(\chi, s) = (\Omega^2(\chi, s)/c_s^2(\chi)) (\partial r^2/\partial s) \).

No further progress looks possible, unless we make a further assumption: \( \Omega = \Omega(\chi) \), i.e ideal MHD! Then we get

\( \rho(\chi, s) = \rho_0 \exp \left(-\frac{r_0^2 - r^2}{w^2}\right) \),

where \( \rho_0 \equiv \rho(\chi, s = 0) \), \( r_0 \equiv r(\chi, s = 0) \), \( M_s \equiv u_\theta/c_s \), the Mach number, and the width \( w = r/M_s \).

- Defining \( s = 0 \) be at the symmetry plane \( \zeta = \pi \), we see that the mass density decays exponentially for \( s > 0 \), if \( M_s \gg 1 \).

- Thus, centrifugal confinement becomes effective only for supersonic rotations.
General observations on equilibrium: 

*limits on rotation velocity*

- $M_s = u_\theta / c_s \gg 1$ is necessary for centrifugal confinement. Then what is the upper limit on $u_\theta$?

- At the symmetry plane, $\zeta = \pi$, radial force balance requires:

- $\partial P/\partial r = \rho r \Omega^2$, where $P \equiv p + B_\zeta^2/2$. Assuming rigid body rotation and integrating leads to

- $u_\theta^2(a) < v_A^2 - c_s^2$, where $v_A = B_\zeta^2(a)/\bar{\rho}$, and $c_s^2 = 2p(0)/\bar{\rho}$.

- For faster rotation, the plasma "flies apart."

- Thus, for effective centrifugal confinement, we need $1 < M_s^2 < (1 - \beta_0)/\beta_0$, where $\beta_0 = c_s^2/v_A^2$. 
Equilibrium: no rotation

Starting with a 1D theta-pinch equilibrium and applying mirror fields, we numerically “relax” to a 2D equilibrium, using an initial value code.
Various profiles:

- For $a$, $b$, and $c$:
  \[ \Omega(\chi) = \Omega_0 \left( 1 - \exp \left( -\frac{(1 - x)^2}{w^2} \right) \right) / \left( 1 - \exp \left( -1/w^2 \right) \right) , \text{where } x \equiv \chi/\chi_{max}. \]
  - $a$) $w = 0.273$, $b$) $w = 0.546$, $c$) $w = 1.09$.

- For $d$:
  \[ \Omega(\chi) = \Omega_0 (1 - x^\lambda)^2 , \text{with } \lambda = 0.6. \]
An equilibrium with medium shear for \( \Omega_0 = 0.73 \) [Profile (b)]
Evolution of a high-shear equilibrium with increasing frequency [Profile (d)]
Low-shear equilibrium for $\Omega_0 = 0.525$ [Profile (a)] - at the radial equilibrium limit

- $\partial P/\partial r = \rho r \Omega^2$, where $P \equiv p + B_\zeta^2/2$, which leads to
- $u_{max}^2(r_1) = (v_A^2(r_1) - v_A^2(0)) + (c_s^2(r_1) - c_s^2(0))$
- Using $r_1 = 0.7$, $\bar{\rho} = 0.75$, $B(r_1) = 0.38$, $B(0) = 0.18$,
- $p(r_1) = 0.024$, $p(0) = 0.035$ leads to
- $u_{max} = 0.35 \sim u_\theta(r_1)$!
- We are at (over?) the radial equilibrium limit!
Linear stability with no rotation

Interchange growth rates

As expected, the mirror equilibrium without rotation is strongly unstable to interchange (flute) modes.

All m’s are expected to be unstable.

Viscous and other dissipative terms in the system help “damp” high mode numbers.

We typically examine modes with m=1,2,3 … 9.
Pressure eigenfunction for $m=6$ and $\Omega=0$. 
Growth rate and real frequency with rotation: low shear [Profile (a)]

- The interchange modes that exist for $\Omega = 0$ are never stabilized.
- This “rigid-rotation” profile is clearly very destabilizing, as expected.
- “Shear stabilization” is more effective for higher mode numbers and occurs very early, long before centrifugal forces become relevant.
- The real frequency scales linearly with the rotation frequency: $\omega_r \sim m\Omega$.
- There is a competing mode with $\omega_r \sim 0$!
Growth rate and real frequency for medium-shear
[Profile (b)]

- With “shear stabilization”, some of the modes are replaced by a zero-frequency mode for $0.1 < \Omega < 0.3$.
- These new modes have only a very weak dependence on rotation frequency.
- For $\Omega > 0.3$, the interchange modes reappear.
Eigenfunction differences between the “zero-frequency” and interchange modes.

- The “zero-frequency” mode typically has $\omega_r/\gamma \sim 10^{-2}$, and $\omega_r/\Omega_0 \sim 10^{-3}$.
- Unlike the interchange mode, it has finite $k_\parallel$.
- It is “localized” to the edge.
- For the mode shown ($m = 8$), we have $\omega_1 = (4.95 \times 10^{-2}, \sim 0)$, and $\omega_2 = (8.60 \times 10^{-2}, 2.19)$ at $\Omega_0 = 0.3$
- The simultaneous presence of these two(?) modes sometimes makes it difficult to converge to an eigenvalue.
Eigenfunction differences between the “zero-frequency” and interchange modes.

- The “zero-frequency” mode is due to a “Parker instability.”

- It is a current driven mode with a finite parallel mode number.

- It can be understood in terms of an “attraction” between current filaments on a sinusoidally perturbed surface.
Growth rate and real frequency for high-shear [Profile (d)]

- The interchange mode is quickly “shear-stabilized.”
- The “zero-frequency” mode persists for all $\Omega$.
- Not stable at any speed!
Straight theta-pincho stability (no mirror fields)

- In order to better understand some of the rotating mirror stability results, here we remove the mirror fields and look at a straight theta-pincho with rotation.
- The problem is simplified by the additional symmetry.
- Since the mirror field itself has a destabilizing influence the results in its absence will tend to overestimate stability (or underestimate the linear growth rates) for the observed modes.

Rotation frequency profiles will be parametrized as:

- $\Omega(x) = \Omega_0 \left(1 - x^\lambda\right)^2$, or
- $\Omega(x) = C\Omega_0 x^{\lambda_1} (1 - x)^{\lambda_2}$, where $x \equiv \chi/\chi_{max}$, and $\chi$ is the toroidal flux function.

We will vary plasma $\beta$, and also look at the effects of an inverted density profile.
Straight $\theta$–pinch growth rates for: $\lambda = 2 \quad \beta = 10\%$

\[ \Omega = 0.10 \]

\[ \Omega(r) \quad \Omega_0 = 0.1 \]

\[ \rho \]

\[ r \]

\[ m \]

\[ n \]
Straight $\theta$–pinch growth rates for: $\lambda = 2 \quad \beta = 10\%$
Straight $\theta$–pinch growth rates for:\n
$$\Omega = 0.1$$  \n
$$\Omega = 0.3$$  \n
$$\Omega = 0.4$$
Straight $\theta$-pinch growth rates for: $\lambda = 2$ $\beta = 10\%$
Straight $\theta$–pinch growth rates for: $\lambda = 0.6 \quad \beta = 10\%$
Straight $\theta$–pinch growth rates for: $\lambda = 2 \quad \beta = 1\%$

$\Omega = 0.1$

$\Omega = 0.15$

$\Omega = 0.25$

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Straight $\theta$–pinch growth rates for: $\lambda = 0.6 \quad \beta = 1\%$
Straight $\theta$–pinch growth rates for: $\Omega(0) = \Omega(1)=0$

$$\Omega(x) = C\Omega_0 x^{\lambda_1} (1 - x)^{\lambda_2},$$
$$\lambda_1 = 1, \quad \lambda_2 = 3, \quad x \equiv \chi/\chi_{max}$$
Straight $\theta$–pinch growth rates for: $\Omega(0) = \Omega(1) = 0$

- $\Omega = 0.20$
- $\Omega = 0.3$
- $\Omega = 0.4$
- $\Omega = 0.475$
Straight θ–pinch growth rates for: $\partial \rho / \partial r > 0$.

- Examine the effects of an “inverted” density profile.
Straight $\theta$–pinch growth rates for: $\frac{\partial \rho}{\partial r} > 0$. 

$\Omega=0.2$ 

$\Omega=0.3$ 

$\Omega=0.4$ 

$\Omega=0.475$
Summary and Future Work

• We are continuing our study of centrifugal confinement schemes.
• With a single-fluid MHD model, we have looked at equilibrium and stability of a limited but typical set of configurations based on a rotating mirror geometry.
• Equilibrium calculations lead to “semi-detached” states that are at least partially insulated from the end-points. However, parallel thermal conductivity still connects the center to the ends.
• Non-rotating mirror equilibria are strongly unstable to ideal MHD modes.
  – Without sufficient shear, these modes are made even more unstable by rotation (interchange modes).
  – With high rotation-shear, they are replaced by “zero-frequency” (Parker) modes (for m=0), and others with finite (m,n), still raising serious questions of accessibility.
• Nonlinear calculations are contemplated.