Finite Larmor Radius Stabilization of Ideal Ballooning Instabilities in 3-D Plasmas
Motivation

- The nature of ideal MHD ballooning modes in 3-D systems differs qualitatively from ballooning modes in 2-D systems
  - Field-line dependence of ballooning mode eigenvalues
  - This typically corresponds to a global mode that is highly localized on the magnetic surface ~ Can nonideal physics (e.g. FLR physics) more easily stabilize these localized modes in 3-D relative to 2-D systems?
  - This work, include FLR effects in ballooning mode formalism of 3-D systems
Ideal MHD ordering and WKB-like formalism is used throughout the field line, the ballooning equation leads to an ordinary differential equation for leading order solution.

For ideal MHD ballooning modes

\[
0 = \mathbf{r} \cdot \mathbf{B} = \mathbf{S} \Delta \mathbf{B}
\]

\[
\frac{\partial (x) \Sigma}{\partial (x) S} = (x) \Sigma
\]

Use large \( k \) expansion

\[
0 = \Sigma \cdot (\mathcal{A} + \mathcal{I} \partial_\zeta \omega)
\]

For ideal MHD ballooning modes
\[
\frac{x^Q}{S^Q} = x^y \quad \frac{b^Q}{S^Q} = b^y \quad x^Q \Delta^y = b^Q \Delta^y = \tau^k
\]

\[
\tau^y = \tau^\theta
\]

Surface label \equiv \theta

Field line label \equiv \alpha

where

\[
\left( \tau^\theta, \mathcal{B}, \alpha \right) \chi = \tau^\alpha
\]

Line and orientation

Solved along each field line for all \( \tau^k \) to find "most unstable field"

\[
\Delta \mathcal{M} \left( \frac{\mathcal{B}}{z^k} \right) = \tau^k \cdot \Delta \mathcal{B} \left( \frac{\mathcal{B}}{z^k} \right) = \tau^k \cdot \Delta \mathcal{B} \left( \frac{\mathcal{B}}{z^k} \right)
\]

Equation of motion to order \( \mathcal{O} \) of "ballpointing equation"

Ballpointing equation
Two-fluid physics brings in finite Larmor radius effects.

- Modified MHD equations modified by Hall-MHD terms in Ohm's law and gyroviscosity.

\[
\sum (i \cdot \nabla \boldsymbol{\gamma} - \omega) \omega \frac{\partial \boldsymbol{B}}{\partial t} = - \sum (\Delta \cdot \boldsymbol{B}) \frac{\partial \boldsymbol{B}}{\partial t} \frac{\partial \boldsymbol{B}}{\partial t} = \sum (\Delta \cdot \boldsymbol{B}) \frac{\partial \boldsymbol{B}}{\partial t} \frac{\partial \boldsymbol{B}}{\partial t}
\]

Modified ballooning equation

\[
i \cdot \nabla \nabla \frac{\partial \boldsymbol{B}}{\partial \Delta} \frac{\partial \boldsymbol{B}}{\partial \Delta} = \sum \frac{\partial \boldsymbol{B}}{\partial \Delta} \frac{\partial \boldsymbol{B}}{\partial \Delta} = \sum (i \cdot \nabla \boldsymbol{\gamma} - \omega) \omega = \sum (i \cdot \nabla \boldsymbol{\gamma} - \omega) \omega = \sum (i \cdot \nabla \boldsymbol{\gamma} - \omega) \omega
\]

Order such that FLR corrections enter

\[
i \cdot \nabla \nabla \frac{\partial \boldsymbol{B}}{\partial \Delta} \frac{\partial \boldsymbol{B}}{\partial \Delta} = \sum \frac{\partial \boldsymbol{B}}{\partial \Delta} \frac{\partial \boldsymbol{B}}{\partial \Delta} = \sum (i \cdot \nabla \boldsymbol{\gamma} - \omega) \omega = \sum (i \cdot \nabla \boldsymbol{\gamma} - \omega) \omega
\]

\[
i \cdot \nabla \nabla \frac{\partial \boldsymbol{B}}{\partial \Delta} \frac{\partial \boldsymbol{B}}{\partial \Delta} = \sum \frac{\partial \boldsymbol{B}}{\partial \Delta} \frac{\partial \boldsymbol{B}}{\partial \Delta} = \sum (i \cdot \nabla \boldsymbol{\gamma} - \omega) \omega = \sum (i \cdot \nabla \boldsymbol{\gamma} - \omega) \omega
\]

Gyroviscosity

\[
i \cdot \nabla \nabla \frac{\partial \boldsymbol{B}}{\partial \Delta} \frac{\partial \boldsymbol{B}}{\partial \Delta} = \sum \frac{\partial \boldsymbol{B}}{\partial \Delta} \frac{\partial \boldsymbol{B}}{\partial \Delta} = \sum (i \cdot \nabla \boldsymbol{\gamma} - \omega) \omega = \sum (i \cdot \nabla \boldsymbol{\gamma} - \omega) \omega
\]

MHD equations modified by Hall-MHD terms in Ohm's law and gyroviscosity effects
\[
\begin{align*}
(\gamma_\theta, b, \omega) \gamma &= \mathcal{U} \\
(\gamma_\theta, b, \omega) \gamma &= \mathcal{M}
\end{align*}
\]

so BE eigenvalue problem same for ideal and non-ideal cases

\[
(\gamma_\theta \omega \alpha \gamma - \mathcal{M}) \mathcal{M} = \mathcal{U}
\]

Define

\[
\mathcal{Z}(\gamma_\theta \omega \alpha \gamma - \mathcal{M}) \mathcal{M} \begin{pmatrix} V \omega \\ T \Omega \end{pmatrix} \mathcal{Z} = \mathcal{Z}(\gamma_\theta \omega \alpha \gamma - \mathcal{M}) \mathcal{M} \begin{pmatrix} V \omega \\ T \Omega \end{pmatrix} \mathcal{Z} = \mathcal{Z}(\gamma_\theta \omega \alpha \gamma - \mathcal{M}) \mathcal{M} \begin{pmatrix} V \omega \\ T \Omega \end{pmatrix} \mathcal{Z} = \mathcal{Z}(\gamma_\theta \omega \alpha \gamma - \mathcal{M}) \mathcal{M} \begin{pmatrix} V \omega \\ T \Omega \end{pmatrix} \mathcal{Z} = \mathcal{Z}(\gamma_\theta \omega \alpha \gamma - \mathcal{M}) \mathcal{M} \begin{pmatrix} V \omega \\ T \Omega \end{pmatrix} \mathcal{Z}
\]

Differ only in right hand sides

Ideal and non-ideal equations
modes
that obey physical quantization rules as
compute action integrals
trace rays of constant \( m'^2 \)

modes [\text{Schrodinger equation}]

To quantize apply 
semi-classical methods (i.e., classical methods)

\( \chi(\alpha, \theta, \phi) \) correspond to a quantizable mode

\( m'^2 \)
If system is integrable, phase space has torus structure.

\[
\frac{\partial \phi}{\partial \epsilon} = \epsilon \mu \\
\frac{\partial \psi}{\partial \epsilon} = \epsilon \nu
\]

\[
\frac{\partial \phi}{\partial \epsilon} = \epsilon \mu \\
\frac{\partial \psi}{\partial \epsilon} = \epsilon \nu
\]

Rays of constant \( m \) obey \( (\epsilon, \epsilon, \epsilon, \epsilon, \epsilon, \epsilon, \epsilon) \leq m \).
around the and contours \( b \) and \( x \) of

\[
bp \cdot S \Delta \int_{b}^{0b} = bp \cdot d \int_{b}^{0b} = S
\]

consider the "action"

and

\( (0d, 0b) \) (\( \varepsilon \) mode by picking and)

Choose a candidate "mode" \( \varepsilon \) and

\[
(b, y) = d \quad \text{and} \quad (b', x) = b
\]

Let

\[\text{Action}\]
Modes correspond to quantizable action integrals.

- Action integrals of WKB trajectories are quantized.

\[ \nu(1 + \nu u) = h p^{\nu} \int_{-1}^{1} \frac{3}{1} \]

- Modes correspond to quantizable action integrals.
The inclusion of FLR physics in 2-D systems is straightforward.

In tokamaks, the local eigenvalue $\chi > 0$, stability is obtained if

$$0 < \chi + \frac{i* \partial \psi}{\partial \zeta} \gamma$$

is conserved. Hence, $\omega = \gamma \gamma + \frac{i* \partial \psi}{\partial \zeta} \gamma = \omega$. Hence, $\omega = \frac{\partial \psi}{\partial \zeta}$ is conserved on WKB orbits.

Eigenvalues are independent of field line label, $k_0$ is a good quantum number. Local toroidal mode number is a good quantum number.

In tokamaks, the $I_0 = \int k_0 \gamma$ quantization is trivial. ---

For unstable local eigenvalue $\chi < 0$, stability is obtained if the criterion is satisfied

$$\int \frac{i* \partial \psi}{\partial \zeta} \gamma = \omega$$

(Tang et al., 1980)
In 3-D systems, the inclusion of FLR physics introduces complications.

- In stellarators, local eigenvalues are generally functions of field lines, $\lambda = \lambda(\psi, \theta, k, \alpha)$, and $\lambda$ and $\gamma$ are no longer constants on WKB rays. (Nevins and Pearlstein, '88)

- Given unstable mode ($\gamma > 0$) described by particular values ($\alpha_o, q_o, k_o$), if mode is stabilized:
  \[
  \dot{\gamma} + \frac{\gamma^2}{\omega^*} + \frac{\omega^*}{\gamma^2} = 0
  \]

- Only the $\alpha$ ray equation changes.

In stellarators, local eigenvalues are generally functions of field lines, $\gamma = \gamma(\psi, \theta, \phi, k, \omega^*)$ and $\lambda$ and $\gamma$ are no longer constants on WKB rays. (Nevins and Pearlstein, '88)
\( 0^\theta \leq \text{fast dependence} \leq \text{eigenvalue calculations (Hudson and Hegna Pop submitted)} \)

Pick a "toy \( \chi \)" to emulate what is seen in stellarator ballooning

- 3-d toy model
$3.1 = b$

$(\eta_\theta b + \kappa) u \cos \alpha + \eta_\theta m \cos \omega + \frac{(\eta b - b)}{c} = (\eta_\theta, \eta \kappa) \chi$

$260.0 = \omega$

$\eta_\theta b + \kappa$ and $\eta_\theta$ must be periodic in $\eta_\theta$.

$(\eta_\theta, \eta \kappa) \chi$
Ideal ray orbits lie on topological spheroids in phase space labeled by $(q, \theta, k)$. 
Marginally stable

Choose some \((\alpha_0, \beta_0, \gamma_0)\) with \(\alpha_0 > 0\), such that mode is

where neither \(\alpha_0\) nor \(\lambda\) are constant

\[0 \leq \lambda + \frac{\nu \gamma}{\nu + \lambda}\]

For a stable mode require

\[
\frac{b \lambda}{\gamma} - = \nu \gamma
\]

\[
\frac{\nu \gamma}{\gamma} = \nu \gamma
\]

\[
\frac{b \lambda \gamma}{\gamma} = b
\]

\[
\frac{\nu \gamma}{\gamma} = \nu
\]

Ray equations for constant \(\omega\)

"stabilization"
The projection of the ray equations into $k$-$\alpha$ space shows closed orbits - quantizable action.
The projection of the ray equations into $kq$ space shows multiple "timescales" $\frac{b}{k}$. The projection of the ray equations into $kq$ space shows multiple "timescales" $\frac{b}{k}$. The projection of the ray equations into $kq$ space shows multiple "timescales" $\frac{b}{k}$. The projection of the ray equations into $kq$ space shows multiple "timescales" $\frac{b}{k}$. The projection of the ray equations into $kq$ space shows multiple "timescales" $\frac{b}{k}$.
approximate integrability of the system
Inclusion of FLR eliminates the topological spheroïds --- integrability?
Summary

- Inclusion of FLR effects into ideal MHD ballooning modes discretizes the spectrum.
- The inclusion of FLR physics on ballooning stability is complicated by the non-constancy of \( \omega \sim k^\alpha \) along periodic ray orbits. \( n \) is not a good quantum number.

- FLR stabilization is given by the criterion:

\[
0 < \gamma^o + \delta \gamma^\max |k^\alpha^o| > 0
\]

- Ray equations ("n" is not a good quantum number)

- The inclusion of FLR physics on ballooning stability is complicated by the non-constancy of \( \omega \sim k^\alpha \) along the ray equations.

- Inclusion of FLR effects into ideal MHD ballooning modes discretizes the spectrum.