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2009 Nucl. Fusion 49 035008

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Negative energy waves and MHD stability of rotating plasmas

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Received 31 October 2008, accepted for publication 11 February 2009

Published 26 February 2009

Online at stacks.iop.org/NF/49/035008

Abstract

Eigenmode analysis of ideal magnetohydrodynamic (MHD) systems with flows is performed. It is shown that the energy of stable oscillatory modes (waves) can be both positive and negative. Negative energy waves always correspond to non-symmetric modes which are nonuniform along the direction of the flow. Coupling of negative and positive energy waves is shown to be a universal mechanism of non-symmetric MHD instabilities in flowing media. To study the stability of non-symmetric modes, a new variational approach is developed based on Lyapunov theory. This approach provides a sufficient and (under some assumptions) necessary stability condition. Specific examples are given to illustrate the developed approach.

PACS numbers: 47.20.Ft, 47.35.Tv, 52.30.Cv, 52.35.Py

1. Introduction

Stability study of rotating plasmas is of great current interest in many applied and fundamental problems. Rotation is a common phenomenon in modern fusion experimental devices (such as tokamaks) where it is believed to stabilize kink and resistive wall modes and suppress turbulence [1]. Plasma rotation plays an important role in forming regimes with improved confinement in tokamaks [2]. At the same time, plasma rotation in the presence of a magnetic field may lead to destabilizing effects, for example, magnetorotational instability (MRI) which is widely accepted now as a source of turbulence and angular momentum transport in accretion discs [3].

The behaviour of many plasma systems is well described by ideal magnetohydrodynamics (MHD). The majority of stability studies in MHD are related to the spectral method—analysis of the eigenvalues of the dynamic operator linearized near the equilibrium state. The methodological difficulty of correct spectral stability analysis is in the necessity of finding not only the eigenvalues of the linearized system but also the corresponding eigenvectors, which have to satisfy the particular boundary conditions. Besides, in the case of systems with plasma flows the linearized operator of dynamics becomes non-Hermitian (non-self-adjoint), therefore its eigenvalues are generally complex [4]. As a result, the spectral stability study

of such systems constitutes a very challenging mathematical problem and is often restricted to simple geometries.

Another way to make a judgement about the stability of the equilibrium state is to use variational methods, e.g. Lyapunov theory. According to the Lyapunov stability theorem the stability of the equilibrium state of a dynamical system is guaranteed if there is a Lyapunov functional—an integral of motion which has a strict local minimum (maximum) at the equilibrium state. There is no regular way to construct a Lyapunov functional. For conservative systems (such as ideal MHD) a natural Lyapunov functional candidate is the total energy of the system. This choice results in the well-known energy principle, first realized in [5] for static MHD equilibrium: if the change in potential energy is positive for any small deviations of a conservative system from the equilibrium state, then such an equilibrium state is stable. From a practical point of view it is important that in the case of a static equilibrium the energy principle gives a condition which is both sufficient and necessary (i.e. a criterion) not only for spectral [6] but also for nonlinear stability [7].

Note for clarity that the perturbation of potential energy cannot be made strictly positive definite in terms of general plasma displacement—there are always nontrivial neutral displacements that do not change the potential energy. In static MHD configurations with nested magnetic surfaces, such

displacements are covered by fluid element relabellings and do not perturb any physical quantity [7]. Therefore, they can be taken out of the analysis. There is no analogous statement for MHD equilibria with flows. In this case, positive semi-definiteness of perturbed potential energy still assures the spectral stability (the absence of exponentially growing perturbations) but does not provide a conclusion about the possibility of perturbations growing slower.

In contrast to the static case, in the presence of stationary plasma flow the energy principle gives only the sufficient stability condition, which is normally too restrictive and can almost never be satisfied [8]. Limited applicability of the energy principle to MHD systems with flows can be explained by the existence in such systems of negative energy waves (NEWs)—stable oscillatory eigenmodes, excitation of which leads to a decrease in the total energy of the system [9]. Different attempts have been made to find a variational approach which generalizes the energy principle for the systems with flows [8, 10–13]; however, this problem is still far from a complete solution. The approach proposed in this paper is an extension of the variational method developed in [13].

The structure of the paper is as follows. In section 2, we investigate the energy of waves in a linearized MHD system and point out the important role of NEWs in the instabilities of plasma flows. In section 3, we consider as an example a system which can be unstable with respect to MRI. In section 4, we suggest a variational approach suitable for the stability study of flowing plasma. Our approach is based on the construction of a Lyapunov functional for linearized MHD systems, which is usually referred to as formal stability analysis. In section 5, the potential applications of our approach are discussed.

2. Energy of eigenmodes in ideal MHD

A lot of important physical information can be revealed from the analysis of the energy of eigenmodes in the framework of ideal MHD. To do this, we start from the well-known linearized dynamic equation for plasma displacement vector ξ [4],

$$\rho\ddot{\xi} + 2\rho(\mathbf{V} \cdot \nabla)\dot{\xi} - \mathbf{F}(\xi) = 0, \quad (1)$$

where the dot denotes a partial time derivative, ρ and \mathbf{V} are stationary values of fluid density and velocity, respectively. The general form of linearized force operator $\mathbf{F}(\xi)$ in ideal MHD is

$$\begin{aligned} \mathbf{F}(\xi) = & -\rho(\mathbf{V} \cdot \nabla)^2 \xi + \rho(\xi \cdot \nabla)(\mathbf{V} \cdot \nabla)\mathbf{V} + \nabla \cdot (\rho\xi)(\mathbf{V} \cdot \nabla)\mathbf{V} \\ & - \nabla \delta P + \frac{1}{4\pi}(\nabla \times \delta \mathbf{B}) \times \mathbf{B} + \frac{1}{4\pi}(\nabla \times \mathbf{B}) \times \delta \mathbf{B}. \end{aligned}$$

Here, \mathbf{B} is the equilibrium magnetic field and $\delta \mathbf{B} = \nabla \times (\xi \times \mathbf{B})$ is its perturbation. The perturbation of fluid pressure δP can be specified by the thermodynamic properties of the system. For example, if the process is adiabatic with adiabatic index γ then $\delta P = -\xi \cdot \nabla P - \gamma P \nabla \cdot \xi$. In the case of incompressible MHD, such an equation appears to be excessive, instead one has to impose the incompressibility condition $\nabla \cdot \xi = 0$.

A number of formal properties of (1) can be established. Force operator $\mathbf{F}(\xi)$ is Hermitian (self-adjoint) in the following sense:

$$\int \eta \cdot \mathbf{F}(\xi) d^3r = \int \xi \cdot \mathbf{F}(\eta) d^3r, \quad (2)$$

while the second term in (1) is anti-Hermitian:

$$\int \rho \eta \cdot (\mathbf{V} \cdot \nabla) \xi d^3r = - \int \rho \xi \cdot (\mathbf{V} \cdot \nabla) \eta d^3r. \quad (3)$$

Integration in (2) and (3) is performed over the fluid volume under the assumption that displacements at the plasma boundary vanish to avoid consideration of boundary and vacuum region perturbations. In order to take into account the possibility of such instabilities as external kink mode, more general boundary conditions providing properties (2) and (3) can be used.

A normal-mode solution to (1) has the form

$$\xi(\mathbf{r}, t) = \hat{\xi}(\mathbf{r})e^{-i\omega t}. \quad (4)$$

Then (1) leads to the eigenvalue problem

$$\omega^2 \rho \hat{\xi} + 2i\omega \rho (\mathbf{V} \cdot \nabla) \hat{\xi} + \mathbf{F}(\hat{\xi}) = 0.$$

Multiplying this equation by complex conjugate $\hat{\xi}^*$ and integrating over the fluid volume, we arrive at a quadratic equation for eigen-frequency ω ,

$$A\omega^2 - 2B\omega - C = 0, \quad (5)$$

where $A = \int \rho |\hat{\xi}|^2 d^3r$, $B = -i \int \rho \hat{\xi}^* \cdot (\mathbf{V} \cdot \nabla) \hat{\xi} d^3r$ and $C = - \int \hat{\xi}^* \cdot \mathbf{F}(\hat{\xi}) d^3r$ are real by definition. The solution to (5) is

$$\omega = \frac{B + s\sqrt{B^2 + AC}}{A}, \quad (6)$$

where either $s = 1$ or $s = -1$ for a particular eigenmode. Therefore, eigenmode is unstable only if $B^2 + AC < 0$.

The dynamics described by (1) provides conservation of energy

$$E = \frac{1}{2} \int (\rho |\dot{\xi}|^2 - \xi^* \cdot \mathbf{F}(\xi)) d^3r, \quad (7)$$

where the displacement ξ is assumed to be complex. Substituting ξ from (4), we obtain the energy of the eigenmode

$$E = \frac{1}{2} (A|\omega|^2 + C) e^{2\gamma t}, \quad (8)$$

where $\gamma = \text{Im}(\omega)$. Since energy is conserved, E in (8) cannot depend on time and must be equal to zero for any unstable eigenmode with $\gamma \neq 0$. Indeed, it can be easily proved for $B^2 + AC < 0$ by substitution of ω from (6) into (8). The potential energy of the unstable eigenmode decreases with time (becomes more negative) as it is transferred to the exponentially growing kinetic energy, but its total energy stays at zero at any time, $E = 0$. This statement applies to both static equilibria and equilibria with flows.

The energy of the stable eigenmode with $\gamma = 0$ is given by

$$E = s\omega\sqrt{B^2 + AC}, \quad (9)$$

and can be either positive (positive energy wave, PEW) or negative (negative energy wave, NEW). The latter is realized for eigenmodes with $-B^2/A < C < 0$ and $\text{sign}(B) = -s$. All NEWs are non-symmetric modes, i.e. they have spatial dependence along the equilibrium flow, so $B \neq 0$ (eigen-spectrum formed by non-symmetric modes is non-symmetric about the imaginary axis in the complex ω -plane).

As discussed in [9], there is an interval of equilibrium parameters at which non-symmetric modes with positive and negative energies can coexist. This is valid in general for small-amplitude waves in moving media described by a Lagrangian [14]. When the frequencies of two waves with different signs of energy merge (resonance condition), the energy can be transferred from the NEW to the PEW leading to instability. In fact, such coupling of the NEW and the PEW is a universal mechanism of any non-symmetric instability in an ideal MHD system with flow.

The eigenmodes with purely real or purely imaginary eigenvalues, which form an eigen-spectrum symmetric about the origin, are defined here as *symmetric* eigenmodes. In particular, they correspond to static equilibrium or modes without spatial dependence along the equilibrium flow; in these cases $B = 0$. If we change the equilibrium parameters of the stable system, symmetric instability occurs only after two real eigenvalues merge at zero frequency and then become imaginary. Energies of symmetric eigenmodes are never negative, which is why their stability is successfully investigated by the use of the energy principle. In the case of non-symmetric modes, the energy principle fails if the NEW can be excited in the system; therefore, modified approaches should be used. Such approaches are developed below.

3. Magnetorotational instability

Consider the energies and frequencies of eigenmodes of incompressible conducting fluid with uniform density ρ rotating in a homogeneous transverse magnetic field $\mathbf{B} = B_0 \mathbf{e}_z$. The equilibrium velocity profile used in our calculations corresponds to the electrically driven flow in the circular channel and has the form

$$V = r\Omega(r)\mathbf{e}_\varphi, \quad \Omega(r) = \frac{\Omega_1 r_1^2}{r^2} \quad (10)$$

in a cylindrical system of coordinates $\{r, \varphi, z\}$. Here, r_1 and r_2 are the inner and outer radii of the channel (we take $r_2/r_1 = 5$), respectively, and Ω_1 is the angular velocity at r_1 . A detailed stability analysis of such flow has been performed in [9, 15], assuming normal modes in the form $\xi(\mathbf{r}, t) = \xi(r) \exp(-i\omega t + im\phi + ik_z z)$.

In figure 1 the dependences of the frequencies of axisymmetric ($m = 0$) and non-axisymmetric modes ($m = 1$) on the equilibrium parameter Ω_1/ω_A are shown, where ω_A is the Alfvén frequency defined as

$$\omega_A = \frac{k_z B_0}{\sqrt{4\pi\rho}}.$$

In this example, axisymmetric eigenmodes are symmetric in terms of the eigen-spectrum. These modes have positive energy when they are stable (figure 1(a)). The merging point of two branches in figure 1(a) corresponds to $\Omega_1/\omega_A \approx 2.0$, which is the threshold of MRI for $m = 0$. The nature of axisymmetric MRI is not related to the subject of NEWs and can be explained by the mechanism similar to one of Rayleigh–Taylor instability [16].

For $m = 1$ modes (non-symmetric in terms of the eigen-spectrum, figure 1(b)), both positive and negative energy waves can coexist in the system when $\Omega_1/\omega_A > 1$. The threshold of

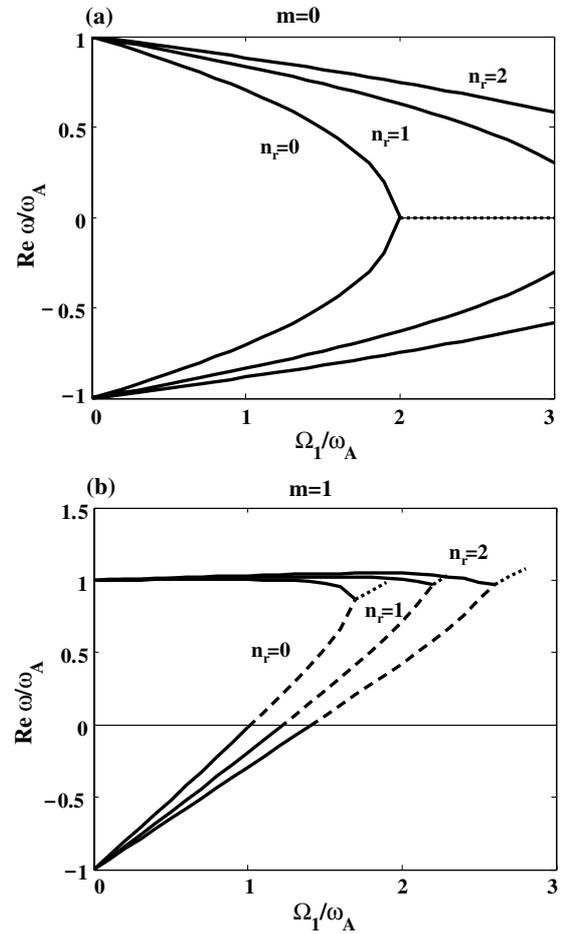


Figure 1. Dependence of eigen-frequencies on the normalized value of angular velocity Ω_1/ω_A for (a) axisymmetric modes with $m = 0$, and (b) non-axisymmetric modes with $m = 1$. Solid lines correspond to PEWs, dashed lines—to NEWs, dotted lines represent the real part of frequency in the unstable region. The value of n_r denotes the radial wave-number, i.e. the number of zeros in the radial direction of the corresponding eigenfunction.

instability in this case is $\Omega_1/\omega_A \approx 1.7$ (it corresponds to the radial mode with $n_r = 0$), when frequencies of the NEW and the PEW are coincident, which is in agreement with the above discussion.

We note that the eigen-spectrum of the system under consideration has a specific property: in the stable region all NEWs have frequencies localized in the interval $\omega \in (0, \omega_0)$ and there are no PEWs in this interval (see figure 1(b)). For the given equilibrium (i.e. for a fixed value of Ω_1/ω_A), the upper boundary ω_0 is always positive; in the case of marginal stability ($\Omega_1/\omega_A \approx 1.7$) it corresponds to the merging point of two branches with radial wave-number $n_r = 0$ ($\omega_0 \approx 0.8\omega_A$).

Such a property of the eigen-spectrum prompts us in an elegant way to modify the energy principle. Suppose we consider perturbations in the reference frame that rotates around the z -axis with constant angular velocity Ω_0 , which is analogous to the following substitution in (1):

$$\xi = \tilde{\xi}(r, z, t) e^{im(\varphi - \Omega_0 t)} \quad (11)$$

(due to axisymmetry of equilibrium we consider perturbations with different m separately). In terms of the eigen-spectrum,

this consideration is equivalent to a shift of all eigenfrequencies according to the rule $\tilde{\omega} = \omega - m\Omega_0$. Choosing $\Omega_0 = \omega_0/m$, where ω_0 is the upper boundary of the NEWs' frequency interval in the laboratory reference frame and $m \neq 0$, we change the sign of frequencies of all NEWs and, therefore, we change the sign of their energies (see (9)). As a result, NEWs are eliminated in the rotating reference frame and the usual energy principle can be applied to establish a sufficient condition for spectral stability.

With substitution (11), the linearized equation of dynamics (1) becomes

$$\rho \ddot{\xi} + 2\rho(\mathbf{V} \cdot \nabla)\dot{\xi} - 2i\omega_0\rho\dot{\xi} - \tilde{\mathbf{F}}(\xi) = 0$$

(we omit the tilde in $\tilde{\xi}$ to simplify the notation). Here $\omega_0 = m\Omega_0$ and $\tilde{\mathbf{F}}(\xi)$ is the 'shifted' force operator:

$$\tilde{\mathbf{F}}(\xi) = \omega_0^2 \xi + 2i\omega_0\rho(\mathbf{V} \cdot \nabla)\xi + \mathbf{F}(\xi).$$

The potential energy of arbitrary displacement ξ in the moving reference frame is

$$\begin{aligned} \tilde{W}(\xi) = & -\frac{1}{2} \int \xi^* \cdot \tilde{\mathbf{F}}(\xi) d^3r = -\frac{1}{2} \int (\omega_0^2 |\xi|^2 \\ & + 2i\omega_0\rho \xi^* \cdot (\mathbf{V} \cdot \nabla)\xi + \xi^* \cdot \mathbf{F}(\xi)) d^3r. \end{aligned} \quad (12)$$

Since ω_0 is not known *a priori*, we reformulate the energy principle in the following theorem, taking into account the presence of neutral eigenmodes in the system—modes with $\tilde{\omega} = 0$, which may be linearly unstable but do not affect the spectral stability.

Theorem 1. *If there exists such real ω_0 that the potential energy (12) is positive semi-definite, i.e. if $\tilde{W}(\xi) \geq 0$ for any ξ , then the equilibrium state is spectrally stable.*

We use this theorem to investigate the stability of the system described above. In this case the stability condition is

$$\begin{aligned} \int_{r_1}^{r_2} ((\omega_A^2 - \tilde{\omega}^2)|\xi|^2 + 2i\tilde{\omega}\Omega(\xi_r^* \xi_\phi - \xi_r \xi_\phi^*) \\ + r(\Omega^2)'|\xi_r|^2) r dr \geq 0, \end{aligned} \quad (13)$$

where $\tilde{\omega} = \omega_0 - m\Omega(r)$ and the prime denotes the radial derivative $\partial/\partial r$.

First, let us examine the stability of axisymmetric perturbations. We have $m = 0$ and $\Omega_0 = 0$; therefore (13) after minimization over ξ_ϕ gives

$$\int_{r_1}^{r_2} \left(\omega_A^2 \left(|\xi_r|^2 + \frac{1}{k_z^2 r^2} |(r\xi_r)')|^2 \right) + r(\Omega^2)'|\xi_r|^2 \right) r dr \geq 0, \quad (14)$$

where the incompressibility condition $\nabla \cdot \xi = 0$ has been used. Condition (14) is a classical result of Chandrasekhar [17], which states that the rotation in an axial magnetic field is stable for perturbations with $m = 0$ if the angular speed, $|\Omega(r)|$, is a non-decreasing function of radius r .

We consider now the stability of rotation profile (10) with respect to perturbations with given azimuthal number $m \neq 0$. Since a conditional minimum is not less than the absolute minimum, we strengthen inequality (13) considering all components of ξ as independent ones (in incompressible

fluid they are related by $\nabla \cdot \xi = 0$). Minimizing (13) over ξ_z and ξ_ϕ we arrive at the condition

$$\begin{aligned} (\omega_A^2 - \tilde{\omega}^2) \geq & \frac{1}{2} (-[r(\Omega^2)'] + 4\Omega^2) \\ & + \sqrt{[r(\Omega^2)'] + 4\Omega^2]^2 + 16\Omega^2\omega_A^2}, \end{aligned} \quad (15)$$

which has to be satisfied at any radius from the interval $r \in [r_1, r_2]$. For the rotation given by (10), the term in square brackets is zero. Then inequality (15) leads to estimates for ω_0 :

$$\begin{aligned} \max_{r \in [r_1, r_2]} (m\Omega(r) - \sqrt{\omega_A^2 - 2\omega_A\Omega(r)}) \\ \leq \omega_0 \leq \min_{r \in [r_1, r_2]} (m\Omega(r) + \sqrt{\omega_A^2 - 2\omega_A\Omega(r)}). \end{aligned}$$

Finally, the stability of the modes with azimuthal number m is guaranteed if

$$\Omega_1 \leq \frac{-4\omega_A + \sqrt{16\omega_A^2 + 4m^2\omega_A^2(1 - r_1^2/r_2^2)^2}}{m^2(1 - r_1^2/r_2^2)^2}. \quad (16)$$

It should be stressed here that in the considered example inequality (13) is also necessary for spectral stability, i.e. if it is not satisfied for some ξ at any ω_0 then the system is unstable. This fact follows from the properties of the eigen-spectrum of the system: (1) all NEWs in the stable region can be eliminated from the eigen-spectrum by simple transition to some moving reference frame; (2) in that reference frame the system does not have neutral eigenmodes (this is true only if $k_z \neq 0$). In general, MHD systems do not possess properties (1) and (2), so the necessary condition for their spectral stability cannot be established by means of theorem 1.

Taking $\xi_r = 0$ in (13) one can conclude that rotation (10) becomes unstable when there exists a radius $r \in [r_1, r_2]$, which makes

$$\omega_A^2 - \tilde{\omega}^2 < 0$$

for any ω_0 . This is satisfied only if

$$\Omega_1 > \frac{2\omega_A}{m(1 - r_1^2/r_2^2)}. \quad (17)$$

From conditions (16) and (17) it follows that at large m the threshold of MRI decreases with m as

$$\Omega_1 = \frac{2\omega_A}{m(1 - r_1^2/r_2^2)}.$$

This result is in agreement with [9, 15].

Theorem 1 can be used only for systems with a very special form of eigen-spectra. In the next section we develop a method for a stability study that works under more general assumptions.

4. Lyapunov stability criterion for plasma flows

It would be ideal to construct an integral of motion of equation (1) which gives a stability criterion (sufficient and necessary condition), i.e. it has a local minimum at the equilibrium state if and only if the equilibrium is stable. In conservative systems a natural first try for such an integral is the energy (7). Treating $\dot{\xi}$ and ξ as independent variables

and minimizing functional $E(\dot{\xi}, \xi)$ over $\dot{\xi}$ which contributes only into non-negative kinetic energy, we arrive at the spectral stability condition in the form

$$W(\xi) = -\frac{1}{2} \int \xi^* \cdot \mathbf{F}(\xi) d^3r \geq 0, \quad (18)$$

which is the well-known energy principle [5].

For static MHD equilibria ($\mathbf{V} = 0$) the energy principle is exhaustive (with reservations mentioned in the introduction), i.e. it gives a stability criterion [6]. In the case of MHD equilibria with flows ($\mathbf{V} \neq 0$), condition (18) originally obtained in [4] is only sufficient for spectral stability. For symmetric perturbations in moving plasmas this condition sometimes appears to be very stiff (far from the necessary one), while for non-symmetric perturbations it may not be satisfied at all if the NEW can be excited in the system.

For MHD equilibria with flows the energy principle can be improved if one takes into account that the displacements $\dot{\xi}$ and ξ in (7) are not completely independent but related by constraints inherent in dynamics (1). In particular, if equation (1) has other invariants (integrals of motion), the local minimum of the energy functional $E(\xi, \dot{\xi})$ has to be established only for a class of displacements that do not change these invariants. Mathematically, these ideas have been formulated by Arnold [18–20] in the following theorem.

Theorem 2 (Arnold). *Let x_0 be an equilibrium point of the system $\dot{x} = f(x)$, i.e. $f(x_0) = 0$. Suppose that the system $\dot{x} = f(x)$ has a set of first integrals (invariants) $E(x), I_1(x), \dots, I_k(x)$. Consider their linear combination:*

$$U(x) = E(x) + \lambda_1 I_1(x) + \dots + \lambda_k I_k(x).$$

Suppose that there exist Lagrange multipliers $\lambda_1, \dots, \lambda_k$ such that:

(1) *the first variation of $U(x)$ is zero at x_0 , i.e.*

$$\delta U(x_0) = \delta E(x_0) + \lambda_1 \delta I_1(x_0) + \dots + \lambda_k \delta I_k(x_0) = 0;$$

(2) *the second variation of $U(x)$ at x_0*

$$\delta^2 U(x_0) = \delta^2 E(x_0) + \lambda_1 \delta^2 I_1(x_0) + \dots + \lambda_k \delta^2 I_k(x_0)$$

is sign-definite on a subspace $\delta I_1(x_0) = 0, \dots, \delta I_k(x_0) = 0$.

Then $U(x)$ is a Lyapunov functional and the equilibrium point x_0 is stable.

It is obvious that the more invariants that are taken into account the closer the Lyapunov stability condition will be to a necessary stability condition. Therefore, Arnold's method is reduced to the search for additional invariants inherent in the dynamical system and analysing conditional extremum of the energy functional.

4.1. Stability of symmetric perturbations

The results of sections 2 and 3 suggest that symmetric and non-symmetric eigenmodes in MHD systems with plasma flows have quite different properties. This allows us to distinguish between symmetric and non-symmetric perturbations which are linear combinations of corresponding eigenmodes. In stability analysis we can consider these two types of

perturbations separately if they are not coupled by equation of dynamics (1), i.e. if any symmetric perturbation evolves independently of any non-symmetric perturbation. Such an assumption is usually valid for simple geometries (as in the example of section 3, where perturbations with different m are independent), but may also be true in more general cases.

We begin our analysis with symmetric perturbations. First, we stress here again that the energy of symmetric eigenmodes is never negative (even if $B \neq 0$ in (5), then $\text{sign}(B) = s$ and $E \geq 0$ in (9)). Since the energy of stable eigenmodes (waves) is additive (theorem 4 in appendix A), the energy of any symmetric perturbation cannot be made negative unless the system has an unstable symmetric eigenmode. This means that positive semi-definiteness of the energy functional gives a spectral stability criterion against symmetric perturbations if all additional constraints for ξ and $\dot{\xi}$ are taken into account.

The additional constraints may be related to other conservative quantities, such as momentum or its components, appearing in the system due to certain geometrical or topological symmetries. The corresponding invariant for equation (1) is expressed in terms of the neutral displacements ξ_N [7, 13]—eigenmodes with $\omega = 0$, satisfying

$$\mathbf{F}(\xi_N) = 0. \quad (19)$$

Indeed, taking the scalar product of equation (1) and ξ_N , integrating over the plasma volume and using definition (19) and property (2), we obtain a conservation law in the form $\dot{I} = 0$, where

$$I = \int (\rho \xi_N \cdot \dot{\xi} + 2\rho \xi_N \cdot (\mathbf{V} \cdot \nabla) \xi) d^3r. \quad (20)$$

According to Arnold's theorem, we have to investigate the sign of the energy functional $E(\dot{\xi}, \xi)$ only for the displacements satisfying $I(\dot{\xi}, \xi) = 0$. Condition $I = 0$, where I is given by (20), can be resolved explicitly for $\dot{\xi}$. Representing $\dot{\xi}$ as

$$\dot{\xi} = -2(\mathbf{V} \cdot \nabla) \xi + \zeta, \quad (21)$$

and substituting (21) in (20), one obtains equation

$$\int \rho \zeta \cdot \xi_N d^3r = 0, \quad (22)$$

which has to be satisfied for every neutral displacement ξ_N . As follows from Fredholm's theorem (theorem 6 in appendix B), this condition is satisfied if and only if

$$\zeta = \frac{\mathbf{F}(\eta)}{\rho}, \quad (23)$$

where $\eta(\mathbf{r})$ is arbitrary vector-function. Therefore, the constraint $I = 0$ yields the local relation between $\dot{\xi}$ and ξ :

$$\dot{\xi} = -2(\mathbf{V} \cdot \nabla) \xi + \frac{\mathbf{F}(\eta)}{\rho}. \quad (24)$$

Substitution of $\dot{\xi}$ from (24) into energy functional (7) gives the following sufficient condition for spectral stability:

$$E_I(\eta, \xi) = \frac{1}{2} \int \left(\frac{1}{\rho} |\mathbf{F}(\eta) - 2\rho(\mathbf{V} \cdot \nabla) \xi|^2 - \xi^* \cdot \mathbf{F}(\xi) \right) d^3r \geq 0. \quad (25)$$

For symmetric perturbations this condition is also necessary for spectral stability if invariant (20) covers all possible linear constraints for $\dot{\xi}$ and ξ . The minimization of the functional $E_I(\eta, \xi)$ over η in systems with nested magnetic surfaces results in the stability condition obtained by different methods in [8, 11, 12]. We note that such minimization leads to the result different from the standard energy principle (18) only if $\rho(\mathbf{V} \cdot \nabla)\xi_N \neq 0$ for some neutral displacement ξ_N . That explains why the sufficient stability condition (14) obtained from the energy principle (18) is also necessary for the stability of symmetric modes (modes with $m = 0$)—the corresponding force operator does not have neutral displacements in that case if $k_z \neq 0$.

4.2. Stability of non-symmetric perturbations

It is clear that for non-symmetric perturbations even the improved energy principle (25) fails if the NEW can be excited in the system. For this reason, other invariants should be involved in the analysis. As shown in [13], the linearized system (1) has an infinite set of exact invariants:

$$E_n = \frac{1}{2} \int (\rho |\xi^{(n+1)}|^2 - \xi^{*(n)} \cdot \mathbf{F}(\xi^{(n)})) d^3r, \quad (26)$$

where $\xi^{(n)}$ is the n th time derivative. Generally, these integrals are independent. E_0 corresponds to the energy (7), higher order invariants (26) have no obvious nonlinear analogues. Using the recurrence relation, which follows immediately from (1),

$$\xi^{(n+2)} = -2(\mathbf{V} \cdot \nabla)\xi^{(n+1)} + \frac{\mathbf{F}(\xi^{(n)})}{\rho},$$

all integrals (26) can be expressed in terms of displacements $\dot{\xi}$ and ξ . In particular,

$$E_1(\dot{\xi}, \xi) = \frac{1}{2} \int \left(\frac{1}{\rho} |\mathbf{F}(\xi) - 2\rho(\mathbf{V} \cdot \nabla)\dot{\xi}|^2 - \dot{\xi}^* \cdot \mathbf{F}(\xi) \right) d^3r,$$

which is similar in structure to invariant E_I (25).

Following Arnold's theorem, we incorporate integrals of motion (26) into a Lyapunov functional candidate by means of Lagrange multipliers λ_n :

$$U(\dot{\xi}, \xi) = \sum_{n=0}^N \lambda_n E_n(\dot{\xi}, \xi). \quad (27)$$

Theorem 3 gives sufficient condition for spectral stability of a system described by (1).

Theorem 3. *If there exist such real numbers λ_n and integer $N \in [0, \infty)$ that $U(\dot{\xi}, \xi) \geq 0$ for all $\dot{\xi}$ and ξ , then form (27) is a Lyapunov functional, and the equilibrium state is spectrally stable.*

Theorem 3 under certain assumptions also provides the necessary condition for spectral stability, i.e. if the system is stable then there are such λ_n which make functional U non-negative for any perturbations.

To demonstrate this, consider a stable system (all eigenvalues ω_j are real). The general solution to (1) is a linear combination of eigenmodes ξ_j

$$\xi = \sum_j c_j \xi_j. \quad (28)$$

For any given initial conditions (i.e. $\xi|_{t=0} = \xi_0$, $\dot{\xi}|_{t=0} = \dot{\xi}_0$), coefficients c_j are uniquely specified. According to theorem 4 (appendix A), integrals (26) for displacement vector (28) can be expressed as

$$E_n(\xi) = \sum_j \omega_j^{2n} |c_j|^2 E(\xi_j),$$

where $E(\xi_j)$ is the energy of the j th eigenmode. Then the value of the Lyapunov functional candidate (27) for displacement (28) is

$$\begin{aligned} U(\xi) &= \sum_{n=0}^N \lambda_n \sum_j \omega_j^{2n} |c_j|^2 E(\xi_j) \\ &= \sum_j |c_j|^2 E(\xi_j) \sum_{n=0}^N \omega_j^{2n} \lambda_n. \end{aligned} \quad (29)$$

In order to make this expression positive for any initial perturbation, we have to ensure that it is positive for every eigenmode independently. This results in conditions

$$\lambda_0 + \omega_k^2 \lambda_1 + \omega_k^4 \lambda_2 + \dots > 0 \quad \text{for every PEW,}$$

$$\lambda_0 + \omega_j^2 \lambda_1 + \omega_j^4 \lambda_2 + \dots < 0 \quad \text{for every NEW.}$$

If these inequalities are satisfied simultaneously by some choice of $\{\lambda_n\}$ then theorem 3 also gives the necessary condition for spectral stability (this is always true for the systems with a countable set of eigenfunctions). We have to emphasize again that (29) is valid for oscillatory modes only. For unstable or decaying modes functional $U(\dot{\xi}, \xi)$ given by (27) cannot be reduced to the form (29) and has no definite sign for any choice of $\{\lambda_n\}$.

It should be noted here that the form (29) cannot be made strictly positive definite if the energy of some eigenmode is zero, i.e. $E(\xi_j) = 0$ for some j . As follows from (9), such a situation is realized either when $\omega_j = 0$ (neutral eigenmode) or when $B_j^2 + A_j C_j = 0$ (marginal stability condition). To separate these two possibilities, one can find the value of potential energy proportional to quantity C from (5). The trial function, which minimizes U at the stability threshold, normally corresponds to $C \neq 0$ for non-symmetric perturbations ($B \neq 0$), while the neutral mode with $\omega = 0$ always provides $C = 0$. There is no such difference for symmetric modes, since $B = C = \omega = 0$ at the stability threshold.

It is evident that theorem 3 also covers the static case and the case of symmetric perturbations. Taking in (27) $U = E_0$, we arrive at the energy principle (18). The choice $U = E_1$ gives the stability condition for symmetric perturbations, which is equivalent to (25).

4.3. Example

To illustrate the developed method, we study the stability of a cold (pressure $P = 0$), constant-density non-magnetized gas rotating around gravitational centre with potential $\Phi(r)$. All equilibrium quantities are assumed to depend only on the radius r in the cylindrical system of coordinates $\{r, \varphi, z\}$. The equilibrium velocity is then

$$\mathbf{V} = r\Omega(r)\mathbf{e}_\varphi, \quad r\Omega^2(r) = \Phi',$$

and the prime denotes a radial derivative $\partial/\partial r$. We look for a solution to (1) in the form of a Fourier series, considering perturbations in the reference frame that rotates around the z -axis with equilibrium angular velocity $\Omega(r)$,

$$\xi(t, \mathbf{r}) = \sum_{m, k_z} \xi_{m, k_z}(t, r) \exp\{im(\varphi - \Omega(r)t) + ik_z z\}.$$

Equation (1) describes the dynamics of each Fourier mode (we omit subscripts)

$$\ddot{\xi} + 2\Omega\dot{\xi} - \hat{B}\xi = 0, \quad (30)$$

where operators \hat{A} and \hat{B} are matrices:

$$\hat{A} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{B} = \begin{pmatrix} -r(\Omega^2)' & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The stability condition for (30) is easily established by the spectral method. Taking $\xi \sim \exp(i\omega t)$, we arrive at the well-known Rayleigh criterion (necessary and sufficient condition for spectral stability)

$$4\Omega^2 + r(\Omega^2)' \geq 0. \quad (31)$$

Now we apply to (30) the developed variational method. First, we note that in the present case all invariants in (26) are local, i.e. corresponding integrands are conserved for every spatial point (r, z) . Second, perturbations described by (30) are independent of m and symmetric—transition to the moving reference frame eliminates all non-symmetric modes from the system. This situation is similar to the example of section 3, where such transition eliminates all NEWs. However, in the present case there are still neutral modes in the system. Therefore, the standard energy principle ($U = E_0$ in (27)) may not give a necessary condition for stability, and modified energy principle ($U = E_1$) should be used. Indeed, the invariants E_0 and E_1 are

$$E_0 = \frac{1}{2}(|\dot{\xi}|^2 - \xi^{*T}\hat{B}\xi) \\ = \frac{1}{2}(|\dot{\xi}_r|^2 + |\dot{\xi}_\varphi|^2 + |\dot{\xi}_z|^2 + r(\Omega^2)'|\dot{\xi}_r|^2),$$

$$E_1 = \frac{1}{2}(|\hat{B}\xi - 2\Omega\dot{\xi}|^2 - \xi^{*T}\hat{B}\xi) \\ = \frac{1}{2}(|r(\Omega^2)'\xi_r - 2\Omega\dot{\xi}_\varphi|^2 + (4\Omega^2 + r(\Omega^2)')|\dot{\xi}_r|^2).$$

If we choose $U = E_1$, we arrive at the stability condition, which is exactly the Rayleigh criterion (31). The standard energy principle ($U = E_0$) gives only the sufficient stability condition $r(\Omega^2)' \geq 0$, which is more restrictive than (31).

In general, if the transition to the moving reference frame cannot eliminate all NEWs in the system, then more than one invariant in (27) has to be used to obtain the stability criterion.

5. Summary

We have demonstrated the physical difference between instabilities of symmetric modes (all modes in static equilibria and modes which are uniform along the equilibrium flows) and non-symmetric modes. Our results show that coupling of two waves with positive (PEW) and negative energies (NEW) can serve as a universal mechanism for any non-symmetric

MHD instabilities of flowing plasma. The energy of symmetric eigenmodes is never negative, so the standard energy principle (18) or its modified version (25) can be successfully applied to study the stability of equilibrium states with respect to symmetric perturbations. However, the energy principle fails for non-symmetric perturbations if NEWs are possible in the system.

To investigate the stability of flowing plasma with respect to non-symmetric perturbations, we have developed a variational method (theorem 3) based on the construction of the Lyapunov functional candidate (27) by incorporating energy with a new set of invariants (26). Under certain assumptions this method can provide a spectral stability criterion (sufficient and necessary condition), at least in the case of discrete eigen-spectra.

The method is verified for a simple analytical example; the obtained stability condition is shown to be both necessary and sufficient. The relative simplicity of the analysis in the considered example is due to the simple form of dynamic operators, which are represented as finite dimensional matrices. In the more general case, to find the adequate stability criterion other integrals from the set (26) have to be included in the analysis.

Acknowledgments

This work is supported in part by the Russian Foundation for Basic Research (project 07-02-00441), the National Science Foundation and the Natural Sciences and Engineering Research Council of Canada.

Appendix A. Additivity of the energy of waves

Theorem 4. *Energy of stable eigenmodes (waves) in linearized ideal MHD is additive.*

Proof. Consider a superposition of two eigenmodes with different real eigenvalues $\omega_1 \neq \omega_2$

$$\xi = c_1\xi_1 + c_2\xi_2.$$

The total energy of this perturbation is

$$E(\xi) = \frac{1}{2} \int (\rho|\dot{\xi}|^2 - \xi^* \cdot \mathbf{F}(\xi)) d^3\mathbf{r} \\ = |c_1|^2 E(\xi_1) + |c_2|^2 E(\xi_2) \\ + \frac{1}{2}c_1c_2^* \int (\omega_1\omega_2\rho\hat{\xi}_1 \cdot \hat{\xi}_2^* - \hat{\xi}_1 \cdot \mathbf{F}(\hat{\xi}_2^*)) d^3\mathbf{r} \\ + \frac{1}{2}c_1^*c_2 \int (\omega_1\omega_2\rho\hat{\xi}_1^* \cdot \hat{\xi}_2 - \hat{\xi}_1^* \cdot \mathbf{F}(\hat{\xi}_2)) d^3\mathbf{r}.$$

The last two integrals in this equation are zero. Indeed, eigenvalue problems for eigenmodes ξ_1 and ξ_2 are

$$\omega_1^2\rho\hat{\xi}_1 + 2i\omega_1\rho(\mathbf{V} \cdot \nabla)\hat{\xi}_1 + \mathbf{F}(\hat{\xi}_1) = 0,$$

$$\omega_2^2\rho\hat{\xi}_2^* - 2i\omega_2\rho(\mathbf{V} \cdot \nabla)\hat{\xi}_2^* + \mathbf{F}(\hat{\xi}_2^*) = 0.$$

Multiplying the first equation by $\omega_2 \hat{\xi}_2^*$ and the second one by $-\omega_1 \hat{\xi}_1$, integrating them over the plasma volume and summing them up, we obtain

$$(\omega_1 - \omega_2) \int (\omega_1 \omega_2 \rho \hat{\xi}_1 \cdot \hat{\xi}_2^* - \hat{\xi}_1 \cdot \mathbf{F}(\hat{\xi}_2^*)) d^3r = 0. \quad (\text{A.1})$$

If $\omega_1 \neq \omega_2$ then the integral in (A.1) is zero. Therefore,

$$E(c_1 \xi_1 + c_2 \xi_2) = |c_1|^2 E(\xi_1) + |c_2|^2 E(\xi_2).$$

This result is easily generalized for any countable number of eigenmodes, q.e.d. An analogous statement can be proved for all integrals of the form (26).

Appendix B. Fredholm's theorem for Hermitian operators

Let L be a linear operator acting on Hilbert space \mathcal{H} —a complete normed space under the norm defined by a scalar product (u, v) for any u and v from \mathcal{H} . We use the following definitions and theorems known from functional analysis [21].

Definition 1 (Hermitian operator). *Linear operator $L: \mathcal{H} \rightarrow \mathcal{H}$ is called a Hermitian (self-adjoint) operator if for any u and v from \mathcal{H}*

$$(v, Lu) = (u, Lv).$$

For Hermitian operators the following theorems hold.

Theorem 5. *If the linear operator L is Hermitian, then all its eigenvalues $\{\lambda_i\}$ are real and its eigenfunctions, corresponding to different eigenvalues, are orthogonal. Moreover, if a set of eigenfunctions is countable, then they form an orthogonal basis in \mathcal{H} .*

Theorem 6 (Fredholm). *The equation*

$$Lu = f, \quad f \in \mathcal{H}, \quad (\text{B.1})$$

where L is a Hermitian operator, can be solved for u if and only if for all v such that $Lv = 0$ the condition $(f, v) = 0$ is satisfied.

Proof. *Necessity.* If equation (B.1) has a solution u and $Lv = 0$, then

$$(f, v) = (Lu, v) = (u, Lv) = 0.$$

Sufficiency. Suppose $(f, v) = 0$ for all v such that $Lv = 0$. According to theorem 5, we can introduce in the space \mathcal{H} orthogonal basis $\{u_k, v_l\}$, where $\{u_k\}$ are the eigenfunctions corresponding to non-zero eigenvalues $\lambda_k \neq 0$ and $\{v_l\}$ are eigenfunctions with zero eigenvalues, i.e. $Lv_l = 0$. It is obvious that the element v can be represented only as a linear combination of $\{v_l\}$. Therefore, in order to satisfy the condition

$(f, v) = 0$, the function f must have the following form

$$f = \sum_k \gamma_k u_k. \quad (\text{B.2})$$

The unknown function u is represented in general as

$$u = \sum_k \alpha_k u_k + \sum_l \beta_l v_l. \quad (\text{B.3})$$

Substitution of expressions (B.2) and (B.3) into equation (B.1) yields

$$\sum_k \alpha_k \lambda_k u_k = \sum_k \gamma_k u_k.$$

Since the eigenfunctions $\{u_k\}$ are orthogonal (theorem 5), we have

$$\alpha_k = \frac{\gamma_k}{\lambda_k}. \quad (\text{B.4})$$

Taking in (B.3) coefficients α_k given by (B.4) and arbitrary β_l , we find the solution to problem (B.1), q.e.d.

In the case considered in section 4, the operator

$$L(\xi) = \frac{\mathbf{F}(\xi)}{\rho} \quad (\text{B.5})$$

is Hermitian (self-adjoint) in terms of the scalar product

$$(\xi, \eta) = \int \rho \xi \cdot \eta d^3r;$$

it follows immediately from property (2). Noting from (22) that $f = \zeta$ and $v = \xi_N$ and applying the Fredholm theorem to operator (B.5), we obtain (23).

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