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Minimum energy states of the cylindrical plasma pinch in single-fluid and Hall magnetohydrodynamics

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Relaxed states of a plasma column are found analytically in single-fluid and Hall magnetohydrodynamics (MHD). We perform complete minimization of the energy with constraints imposed by invariants inherent in the corresponding models. It is shown that the relaxed state in Hall MHD is a force-free magnetic field with uniform axial flow and/or rigid azimuthal rotation. In contrast, the relaxed states in single-fluid MHD are more complex due to the coupling between velocity and magnetic field. Cylindrically and helically symmetric relaxed states are considered for both models. Helical states may be time dependent and analogous to helical waves, propagating on a cylindrically symmetric background. Application of our results to reversed-field pinches (RFP) is discussed. The radial profile of the parallel momentum predicted by the single-fluid MHD relaxation theory is shown to be in reasonable agreement with experimental observation from the Madison symmetric torus RFP experiment. © 2012 American Institute of Physics.

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I. INTRODUCTION

Many magnetized plasma systems exhibit the phenomenon of self-organization—the spontaneous tendency to evolve toward preferred configurations with ordered structure. Theoretical prediction of such configurations is a long-standing problem for both laboratory and astrophysical applications. Due to the complexity and nonlinearity of plasma behavior there is no universal mathematical methodology, except direct numerical simulations, that would be able to describe the final self-organized states in all systems.

Among the plasma systems, whose self-organized states can be predicted theoretically, are the systems without external energy supply. In such isolated systems, the process of self-organization is usually referred to as relaxation. The concept of plasma relaxation was proposed by Taylor, who conjectured that during turbulent dynamics a slightly resistive magnetohydrodynamic (MHD) system tends to minimize its magnetic energy while conserving the total magnetic helicity. The underlying basis of this approach is the principle of selective decay of invariants, i.e., one or more ideal invariants of the system (conserved in the absence of dissipation) are less susceptible to dissipation than energy and thus can be considered as constants during the relaxation process. Mathematically the relaxation theory is formulated as a variational procedure for obtaining a relaxed state by minimizing the energy subject to constraints.

The Taylor theory has been successfully tested in experiments and applied for explaining the magnetic structures in laboratory plasmas such as the reversed-field pinch (RFP), multipinch, and spheromak. However, the classical Taylor theory does not include possibility of flows that are ubiquitous in experimentally observed relaxed states. The origin of these flows is not well understood; laboratory plasmas rotate in the toroidal and poloidal directions even in the absence of externally applied torques (intrinsic rotation). Further, the experimental parameters are such that the single-fluid MHD model may not be strictly valid, and the inclusion of the effects of separate ion and electron fluids in the model may be required.

The present work is motivated by the recent progress in plasma velocity measurements in the Madison symmetric torus RFP experiment, which show an abrupt change of the global flows during the relaxation events. Detailed temporal and spatial measurements of flow dynamics indicate significant radial angular momentum transport and flattening of the radial flow profiles.

The goal of the present paper is to determine the minimum energy (relaxed) states for a cylindrical RFP, to analyze the possibility of plasma flows in such states in both standard (single-fluid) and Hall MHD (a two-fluid model with massless electrons), and to elucidate their global properties. Since the RFP has significant fraction of "bad" magnetic curvature caused by poloidal magnetic field, the effect of the toroidal curvature can be ignored. The geometry of periodic cylinder is a good approximation for RFP theory and simulations. We employ a variational procedure that includes all ideal invariants inherent in corresponding models. While the experiments are open systems that interact with the external environment through applied voltages, here we consider only closed systems. This is consistent with Taylor’s approach, which has been reliable for predicting the general properties of relaxed magnetic fields without flow. The fields and flows predicted by the present theory may be relevant to the flows that are observed after the crash phase of sawtooth cycle in the RFP. Of course, we cannot comment on the origin of these flows, only on their relaxed properties.

In the framework of single-fluid MHD, the problem of finding relaxed states with flows in geometries relevant to

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RFp is addressed in several papers. The relaxed flows in these papers are obtained by including into energy minimization procedure the additional constraints, such as cross helicity and momentum (or its components). It should be emphasized here that the cross helicity is an ideal invariant of incompressible MHD and in some special cases of compressible MHD (e.g., Ref. 8); it is also a rugged invariant during relaxation in the presence of dissipation, which is confirmed by numerical simulations. Energy minimization with the cross helicity constraint in periodic cylinder yields a relaxed state with flow parallel to force-free magnetic field (results are corrected in Ref. 10). In Refs. 8 and 11 in addition to the cross helicity, the total angular momentum is taken into account as a conserved quantity in toroidal geometry. This leads to a relaxed state with combination of parallel flow and rigid toroidal rotation. In present paper, we generalize the results of Refs. 9 and 10 by considering along with the cross helicity the total angular and axial momenta as additional invariants of the cylindrical plasma pinch in incompressible MHD.

The relaxation problem in the framework of Hall MHD is considered in numerous papers. In Refs. 13–20, the use of electron and ion helicities as invariants during relaxation is theoretically substantiated and the double-Beltrami structure of the relaxed states in Hall MHD is revealed. References 13, 14, and 19 give several analytical examples of the relaxed states for different geometries. However, these solutions do not correspond to true minimum energy states for fixed electron and ion helicities as the minimization procedure is not completed. This is because the unknown Lagrange multipliers used in the variational procedure are not specified in terms of the initial values of the invariants. A more complete analysis is reported in Ref. 21, where the energy of relaxed states is found in toroidal systems as a function of electron and ion helicities. Although the general solution of the double-Beltrami equation has two eigenfunctions, Ref. 21 uses only one of them. This precludes the possibility of two different spatial scales in the relaxed state (as in Refs. 19, 22, and 23). In Ref. 22, relaxed states are obtained as a linear combination of two orthogonal Beltrami eigenfunctions with eigenvalues \( \lambda_1 \) and \( \lambda_2 \), respectively, and the energy is expressed as a function of electron and ion helicities and eigenvalues \( \lambda_1 \) and \( \lambda_2 \). The next step is to find a pair of eigenvalues \( (\lambda_1, \lambda_2) \) that minimizes the energy; however, this step is missing in Ref. 22.

The full energy minimization of the incompressible Hall MHD system with fixed electron and ion helicities is completed in Ref. 23 assuming the general geometry and orthogonality of the basis Beltrami vectors (which is not true for cylindrically symmetric states). The result is quite surprising: the relaxed state in Hall MHD is always a force-free magnetic field with no plasma flows, i.e., nothing but the Taylor state. Moreover, the authors of Ref. 23 question the conservation of the ion helicity, arguing that it is not a rugged invariant during relaxation and, therefore, it should not be included into energy minimization procedure. The fact that the ion helicity is not conserved in Hall MHD relaxation is confirmed by numerical simulations. In present paper, we extend the results of Ref. 23 for cylindrical plasma pinch in incompressible Hall MHD assuming the conservation of both electron and ion helicities and total angular and axial momenta. These velocity related invariants allows us to obtain a relaxed Hall state with plasma flows.

One of the novel results of our paper is the prediction of non-stationary relaxed states, which have the form of helical waves propagating on a cylindrically symmetric background. Such states are realized in both single-fluid and Hall MHD when some of velocity related invariants are not zero.

The paper is organized as follows. In Sec. II, the invariants of incompressible single-fluid and Hall MHD models are introduced. In Secs. III and IV, the minimum energy states are obtained for corresponding models. In Sec. V, comparison of these states with experimental observations is performed. In Sec. VI, these results are summarized and their application for reversed-field pinches is discussed.

II. STATEMENT OF THE PROBLEM

We consider the problem of finding the minimum energy (relaxed) states of an axially periodic cylindrical plasma pinch, with a periodicity length \( L \) (Fig. 1). We assume that plasma is incompressible with spatially uniform density \( \rho \), and it is surrounded by a perfectly conducting shell (flux conserver) of radius \( a \). The incompressibility assumption is relevant to RFP physics due to the fact that relaxation in these systems is provided by tearing modes. These modes are slow on Alfvénic time scale and, therefore, essentially incompressible. Moreover, the resulting incompressible relaxed states are in good agreement with experimental observations, as shown below in Sec. V.

Under these assumptions the plasma is described by equations of ideal incompressible Hall MHD, which in non-dimensional form are

\[
\frac{\partial \mathbf{v}}{\partial t} + \nabla \left( p + \frac{\mathbf{v}^2}{2} \right) = \mathbf{v} \times (\nabla \times \mathbf{v}) + (\nabla \times \mathbf{b}) \times \mathbf{b}, \quad \nabla \cdot \mathbf{v} = 0, \quad (1)
\]

\[
\frac{\partial \mathbf{b}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{b} - \epsilon (\nabla \times \mathbf{b}) \times \mathbf{b}), \quad \nabla \cdot \mathbf{b} = 0. \quad (2)
\]

Here, all physical quantities are normalized,

\[
r = \frac{R}{a}, \quad z = \frac{Z}{a}, \quad l = \frac{L}{a}, \quad \tau = \frac{V_A}{a t}, \quad \mathbf{v} = \frac{\mathbf{V}}{V_A}, \quad \mathbf{b} = \frac{\mathbf{B}}{B_0},
\]

\[
p = \frac{p}{\rho V_A^2},
\]

where the Alfvén velocity \( V_A \) and the characteristic magnetic field \( B_0 \) are defined as

![FIG. 1. Geometry of the problem.](image)
with Φ₀ being a total axial magnetic flux, which is constant due to perfectly conducting boundary. Equations (1) and (2) also contain the non-dimensional ion skin depth (or Hall parameter) $\epsilon$, which is defined as

$$\epsilon = \frac{d_i}{a} = \frac{c}{a} \sqrt{\frac{m_i^2}{4\pi\rho_e e^2 Z^2}},$$

(4)

where $m_i$ and $eZ$ are ion mass and charge, $c$ is speed of light. In the limit $\epsilon \rightarrow 0$, single-fluid MHD is recovered.

We adopt cylindrical coordinate system $\{r, \varphi, z\}$ with volume element $d^3r = rdr \; d\varphi \; dz$. In order to solve Eqs. (1) and (2) uniquely, we have to specify boundary conditions for velocity and magnetic field. At an impermeable surface the normal component of electric field are zero at the perfectly conducting boundary. Equations (1) and (2) have been introduced into analysis.\textsuperscript{1,9,11,13,14} In this section, we examine these invariants for cylindrical pinch geometry assuming boundary conditions [Eqs. (5) and (6)]

$$v_r\bigg|_{r=1} = 0,$$

(5)

and the normal component of magnetic field and the tangential component of electric field are zero at the perfectly conducting boundary, which is equivalent to

$$b_r\bigg|_{r=1} = 0, \quad \epsilon j_r\bigg|_{r=1} = 0,$$

(6)

where $j = \nabla \times b$ is normalized current density. Note that in single-fluid MHD ($\epsilon = 0$) the second condition in Eq. (6) is satisfied automatically.

According to the fundamental idea of the Taylor theory, a weakly dissipative system reaches a state of minimum energy without significant change of certain global quantities. These quantities usually correspond to ideal invariants; they are conserved in ideal system and decay slowly (slower than energy) in presence of dissipation. A number of these ideal invariants inherent in system [Eqs. (1) and (2)] have been introduced into analysis.\textsuperscript{1,9,11,13,14} In this section, we examine these invariants for cylindrical pinch geometry assuming boundary conditions [Eqs. (5) and (6)]

$$l_1 \equiv H_e = \int A \cdot \nabla \times A \; d^3r, \quad l_1\bigg|_{r=0} = \pi l K.$$

(8)

Here $A$ is vector potential, such that $b = \nabla \times A$. The time dynamics of vector potential follows from Eq. (2):

$$\frac{\partial A}{\partial t} = -\nabla \phi + v \times b - \epsilon (\nabla \times b) \times b,$$

where $\phi$ is normalized electric potential. As a result, the time derivative of magnetic helicity is

$$\frac{\partial l_1}{\partial t} = -\int_s (\phi b + A \times (v \times b - \epsilon j \times b)) \cdot dS,$$

which is zero if electric potential $\phi$ is periodic in $z$ and boundary conditions [Eqs. (5) and (6)] are satisfied. Therefore, magnetic helicity is an ideal invariant; it can change only in presence of resistivity. For our study it is important that in resistive systems magnetic helicity is more robust than the energy.\textsuperscript{1–3} i.e., we can assume that its value is approximately constant in time and equal to its initial value, $l_1 = \pi l K$, where $\pi l$ is dimensionless volume of the cylinder and $K$ is the volume-averaged density of the magnetic helicity. For unique definition of vector potential and, therefore, magnetic helicity we use a gauge invariance condition\textsuperscript{27}

$$\int_0^1 A_r \; dz = 0.$$

(9)

Single-fluid incompressible MHD system ($\epsilon = 0$) has the well-known ideal invariant–cross helicity,

$$l_2 = \int v \cdot b \; d^3r, \quad l_2\bigg|_{r=0} = \pi l M.$$

(10)

Indeed, its time derivative is

$$\frac{\partial l_2}{\partial t} = -\int_s \left( \left( p + \frac{v^2}{2} \right) b + v \times (v \times b) \right) \cdot dS,$$

which is zero for boundary conditions [Eqs. (5) and (6)]. Cross helicity can be considered also as a rugged invariant (approximately conserved quantity) during relaxation. Its ruggedness in a dissipative single-fluid MHD system is confirmed by numerical simulations for cases, where relaxation is slow on Alfvénic time scales\textsuperscript{12}—typical situation for RFP experiments, in which relaxation is provided by slow tearing modes.\textsuperscript{3} In our single-fluid MHD analysis, we assume that the cross helicity is constant and equal to its initial value, $l_2 = \pi l M$.

The analogue of the cross helicity in incompressible Hall MHD ($\epsilon \neq 0$) is generalized cross helicity,

$$l_2 = \int \left( b + \frac{\epsilon}{2} \nabla \times v \right) d^3r, \quad l_2\bigg|_{r=0} = \pi l M.$$

(11)

It is related to ion helicity

$$H_e = \int (A + \epsilon v) \cdot \nabla \times (A + \epsilon v) d^3r$$

where $A_0$ is the magnetic vector potential, $\phi$ is the electric scalar potential, $\rho$ is the density, $v$ is the velocity, $b$ is the magnetic field, $D_s$ is the stress tensor and $\epsilon$ is the electric conductivity. To avoid any confusion in the notations, in this section we will adopt cylindrical coordinates for the analysis and section.
by the equation
\[ H_i = H_i^0 + 2i\dot{J}_2 + \epsilon \int_S (\mathbf{v} \times \mathbf{A}) \cdot d\mathbf{S}. \]

Their time derivatives are, respectively,
\[ \frac{\partial \dot{J}_2}{\partial \tau} = -\int_S \left[ \mathbf{p} \mathbf{b} + \mathbf{v} \times (\mathbf{v} \times \mathbf{b}) + \frac{\epsilon}{2} (p\mathbf{\omega} + \mathbf{v} \times (\mathbf{v} \times \mathbf{\omega}) \right. \\
- \mathbf{v} \times (\mathbf{j} \times \mathbf{b}) \left. \right] \cdot d\mathbf{S}, \]
\[ \frac{\partial H_i}{\partial \tau} = \int_S \left[ \left( \phi + \epsilon \left( p + \frac{\mathbf{v}^2}{2} \right) \right) (\mathbf{b} + \epsilon \mathbf{\omega}) + (\mathbf{A} + \epsilon \mathbf{\omega}) \right. \\
\times (\mathbf{v} \times (\mathbf{b} + \epsilon \mathbf{\omega})) \left. \right] \cdot d\mathbf{S}, \]

where \( \omega = \nabla \times \mathbf{v} \) is the fluid vorticity. Taking into account boundary conditions [Eqs. (5) and (6)], time derivatives of the generalized cross helicity and ion helicity become
\[ \frac{\partial \dot{J}_2}{\partial \tau} = -\int_S \left( \frac{\mathbf{v}^2}{2} - p \right) \omega \cdot d\mathbf{S}, \]
\[ \frac{\partial H_i}{\partial \tau} = \epsilon \int_S \left( \mathbf{v} \cdot \mathbf{A} - \phi + \epsilon \left( \frac{\mathbf{v}^2}{2} - p \right) \right) \omega \cdot d\mathbf{S}. \]

From these equations one can see that in order for \( \dot{J}_2 \) and \( H_i \) to be conserved, an extra boundary condition must be imposed,\(^\text{14}\) which is
\[ \omega \bigg|_{r=1} = 0. \]

This condition is satisfied automatically for all time if it is satisfied initially at \( \tau = 0 \). This is guaranteed by boundary conditions [Eqs. (5) and (6)] and by equation of ideal dynamics of the fluid vorticity
\[ \frac{\partial \omega}{\partial \tau} = \nabla \times (\mathbf{v} \times \omega + \mathbf{j} \times \mathbf{b}), \]

which is obtained from Eq. (1) by taking the curl. In the following, we assume that the generalized cross helicity (and, hence, the ion helicity) is conserved during relaxation and its value is \( I_2 = \pi lM \). As we show in Sec. IV, relaxed state in incompressible Hall MHD is completely independent of this invariant: energy minimizations with or without generalized cross helicity lead to the same result.

Equations (2) and (13) being similar in structure also guarantee the conservation of the axial fluxes of the magnetic field and the fluid vorticity, respectively,
\[ I_3 = \int_0^{2\pi} \int_0^1 b_3 r dr d\phi, \quad I_3 \bigg|_{r=0} = \pi, \]
\[ I_4 = \int_0^{2\pi} \int_0^1 \omega_4 r dr d\phi = \int_0^{2\pi} v \bigg|_{r=1} d\phi, \quad I_4 \bigg|_{r=0} = 2\pi \Omega_0, \]

where we used normalization [Eq. (3)] and assumed that the azimuthally averaged angular velocity at the boundary is \( \Omega_0 \). Their time derivatives are
\[ \frac{\partial I_3}{\partial \tau} = \int_0^{2\pi} \left( \mathbf{v} \times \mathbf{b} - \mathbf{j} \times \mathbf{b} \right) \cdot d\mathbf{S}, \]
\[ \frac{\partial I_4}{\partial \tau} = \int_0^{2\pi} (\mathbf{v} \times \mathbf{\omega} + \mathbf{j} \times \mathbf{\omega}) \cdot d\mathbf{S}. \]

Note that magnetic flux [Eq. (14)] is constant in both single-fluid and Hall MHD due to boundary conditions [Eqs. (5) and (6)], while fluid vorticity flux [Eq. (15)] is conserved only in Hall MHD when additional condition [Eq. (12)] is satisfied.

The geometrical symmetry of the pinch configuration yields two more ideal invariants, the axial and angular momenta,
\[ I_5 = \int_0^{2\pi} r v_3 d^3 r, \quad I_5 \bigg|_{r=0} = \pi l u, \]
\[ I_6 = \int_0^{2\pi} r v_\phi d^3 r, \quad I_6 \bigg|_{r=0} = \pi l \Omega \frac{l}{2}. \]

Their time derivatives are zeros for boundary conditions [Eqs. (5) and (6)] since
\[ \frac{\partial I_5}{\partial \tau} = \int_0^{2\pi} (\mathbf{b} \times \mathbf{v} - \mathbf{b} \times \mathbf{j}) \cdot d\mathbf{S}, \]
\[ \frac{\partial I_6}{\partial \tau} = \int_0^{2\pi} \left( r \mathbf{b} \times \mathbf{v} - r \mathbf{v} \times \mathbf{b} \right) \cdot d\mathbf{S}. \]

The initial values of these invariants can always be attributed to some uniform flow with velocity \( u \) in \( z \)-direction and a rigid rotation with angular velocity \( \Omega \) in \( \varphi \)-direction.

In the following, we use these invariants as constraints in energy minimization procedure.

III. RELAXED STATES IN SINGLE-FLUID MHD

In this section, we study single-fluid MHD (\( \epsilon = 0 \)) relaxed states. We consider two cases here: the most general case, where relaxed state corresponds to a minimum of energy [Eq. (7)] subject to constraints given by Eqs. (8), (10), (14), (16), and (17); and the case, where we ignore axial and angular momenta constraints given by Eqs. (16) and (17) for the reasons explained below.

A. Energy minimization with full set of invariants

First we consider the case, where relaxed state is found by minimizing energy [Eq. (7)] with the most general set of invariants [Eqs. (8), (10), (14), (16), and (17)] inherent in incompressible single-fluid MHD in periodic cylinder. The Lagrange multipliers method results in variational problem,
\[ \delta \left[ E + \mu_1 (I_1 - \pi l K) + \mu_2 (I_2 - \pi l M) + \mu_3 (I_3 - \pi) \right. \]
\[ + \mu_4 (I_4 - \pi l u) + \mu_5 (I_5 - \pi l u) + \mu_6 \left( I_6 - \frac{\pi l \Omega}{2} \right) \bigg] = 0, \]

which determines a conditional extremum of \( E \) (it is minimum since \( E \) is positive definite, and, therefore, the resulting equilibrium is ideally stable). The Euler equations of variational problem [Eq. (18)] are
\[ \begin{align*}
\sim \delta v : & \quad v_0 + \mu_3 b_0 + \mu_4 e_z + \mu_6^r e_\phi = 0, \\
\sim \delta \lambda : & \quad \nabla \times b_0 + 2\mu_1 b_0 + \mu_2 \nabla \times v_0 = 0, \\
\sim \delta \mu_k : & \quad I_1 = \pi l K, \quad I_2 = \pi l M, \quad I_3 = \pi, \quad I_5 = \pi l u, \\
I_6 &= \pi l \Omega / 2.
\end{align*} \]

Equations (19) and (20) can be reduced to one equation for magnetic field

\[ \nabla \times b_0 - \lambda b_0 = \frac{2\mu_3 \mu_6^r}{1 - \mu_5^2} e_z, \]

where

\[ \lambda = \frac{2\mu_1}{\mu_5^2 - 1}. \]

Thus, the relaxed magnetic field is no longer force-free as in classical Taylor theory due to its coupling with the relaxed flow. Note that the amplitude of the magnetic field \( b_0 \) and the Lagrange multipliers \( \mu_1, \mu_2, \mu_5, \mu_6^r \) are not arbitrary, they are determined by constraints from Eq. (21). Therefore, the relaxed state depends only on the initial values of the invariants.

The most general solution to Eqs. (19) and (20) satisfying gauge invariance condition [Eq. (9)] and constraints \( I_3 = \pi, I_5 = \pi l u, I_6 = \pi l \Omega / 2 \) is a superposition of a cylindrically symmetric mode (with the azimuthal mode number \( m = 0 \) and the axial wave-number \( l_z = 0 \)) and modes with \( m \neq 0 \) or \( k_z \neq 0 \),

\[ \begin{align*}
A_0 &= \frac{r}{2} e_\phi + \frac{C}{\lambda} \left[ (J_1(\lambda r) - J_1(\lambda r)) e_\phi + \left( J_0(\lambda r) - \frac{2J_1(\lambda r)}{\lambda} \right) e_z \right] + \frac{1}{2} H, \\
b_0 &= \nabla \times A_0 = \vec{e}_z + C \left[ \left( J_1(\lambda r) e_\phi + \left( \frac{J_0(\lambda r) - 2J_1(\lambda r)}{\lambda} \right) e_z \right) + H, \\
v_0 &= \Omega e_\phi + u e_z - \mu_5 C \left[ \left( J_1(\lambda r) - \frac{4J_2(\lambda r)}{\lambda} r \right) e_\phi + \left( \frac{J_0(\lambda r) - 2J_1(\lambda r)}{\lambda} \right) e_z \right] + \mu_2 H,
\end{align*} \]

where \( J \) denotes Bessel functions of the first kind, coefficient \( C \) is

\[ C = \frac{\lambda (1 - \mu_5^2) - 2\mu_5^2}{2J_1(\lambda) + 2\mu_2^2 J_2(\lambda)}, \]

and \( H \) represents the helical part of the solution (modes with \( m \neq 0 \) or \( k_z \neq 0 \), such that \( \nabla \times H = \lambda H \). Substituting these expressions into Eqs. (7), (8), and (10), we obtain the volume-averaged densities of the energy, the magnetic helicity and the cross helicity,

\[ \begin{align*}
E = & \frac{1}{2l^2} + \frac{u^2}{l^2} + \frac{\Omega^2}{4} + \frac{1}{2} \frac{\mu_5^2 C^2}{2} \left( \frac{2J_2(\lambda) - 3J_0(\lambda)J_2(\lambda)}{\lambda} - J_2^2(\lambda) \right) - \frac{4\mu_5^2 C^2 J_z(\lambda) + (1 + \mu_5^2) D}{\lambda^2}, \end{align*} \]

where \( D = \frac{2C J_2(\lambda)}{\lambda} + \frac{2C^2}{\lambda} \left( \frac{J_1(\lambda)}{\lambda} - 2J_0(\lambda)J_2(\lambda) - J_2^2(\lambda) \right) + \frac{D}{\lambda}, \)

\[ M = u + \frac{2\Omega C J_2(\lambda)}{\lambda} - \mu_5 C^2 \left( \frac{2J_1(\lambda)}{\lambda} - 3J_0(\lambda)J_2(\lambda) - J_2^2(\lambda) \right) - \frac{8J_2^2(\lambda)}{\lambda^2} - \mu_2 D, \]

where \( D \) is a non-negative quantity characterizing magnitude of the helical part,

\[ D = \frac{1}{\pi l} \int H^2 d^3 r. \]

Consider first the cylindrically symmetric solution with \( D = 0 \). In this case Eqs. (27) and (28) are used to find \( \mu_5 \) and \( \lambda \) through the initial values of helicities, \( K \) and \( M \). The resulting volume-averaged energy density of the relaxed state \( E_{cs} = (E/\pi l)_{D=0} \) as a function of \( K \) is shown in Fig. 2. A sample of the relaxed state with non-zero plasma flows is shown in Fig. 3.

FIG. 2. Volume-averaged energy density of cylindrically symmetric MHD relaxed state [Eqs. (23)–(25)] \( E_{cs} \) as a function of magnetic helicity \( K \) (solid curve) and its difference with energy density of helically distorted MHD relaxed state \( E_{hel} \) in domain of its existence (dashed curve). Note different scales for positive and negative energies. Results are presented for cross helicity \( M = 0.1 \), axial momentum \( u = 0 \), angular momentum \( \Omega = 0 \) and \( \lambda = 3.11 \) for helical state.
helically distorted relaxed states in the domains occupied by cylindrically symmetric and torqued relaxed state is completed by expressing its magnitude that only one value of \( l \) is possible in the relaxed state as can be seen from Eq. (22). The description of the helically distorted relaxed state is completed by expressing its magnitude \( D \) and the Lagrange multiplier \( \mu_2 \) from Eqs. (27) and (28) in terms of the initial values of helicities, \( K \) and \( M \). Our calculations show that in the domain of its existence the helically distorted state has lower energy \( E_{\text{hel}} = (E/\pi l)_D > 0 \) than the cylindrically symmetric state (Fig. 2); besides, the lowest energy is achieved when the value of \( \lambda \) is minimal. This minimal value \( \lambda = 3.11 \) corresponds to helical mode with \( m = 1 \) and \( k_z = 1.23 \) (Fig. 4); in our following analysis we assume that the helical distortion is due to this mode, i.e.,

\[
m\lambda J_m(x) + k_z x J'_m(x) = 0.
\] (31)

For given \( m \) and \( k_z \), this equation determines \( \lambda \) (Fig. 4). Note that only one value of \( \lambda \) is possible in the relaxed state as can be seen from Eq. (22). The description of the helically distorted relaxed state is completed by expressing its magnitude \( D \) and the Lagrange multiplier \( \mu_2 \) from Eqs. (27) and (28) in terms of the initial values of helicities, \( K \) and \( M \). Our calculations show that in the domain of its existence the helically distorted state has lower energy \( E_{\text{hel}} = (E/\pi l)_D > 0 \) than the cylindrically symmetric state (Fig. 2); besides, the lowest energy is achieved when the value of \( \lambda \) is minimal. This minimal value \( \lambda = 3.11 \) corresponds to helical mode with \( m = 1 \) and \( k_z = 1.23 \) (Fig. 4); in our following analysis we assume that the helical distortion is due to this mode, i.e.,

\[
H = C_h \left|_{\lambda = 3.11, \ m = 1, \ k_z = 1.23} \right.
\] (32)

The quantity \( D \) defined in Eq. (29) can be expressed in terms of the amplitude of the helical distortion \( C_h \),

\[
D = \frac{C^2 h^2 J^2_m(x)}{2} \left( 1 + \frac{m^2}{k_z^2} - \frac{m\lambda}{k_z^2} \right).
\] (33)

The domains occupied by cylindrically symmetric and helically distorted relaxed states in the \( K - M \) plane are presented in Fig. 5, the boundary between them can be determined by setting \( D = 0 \) in Eqs. (27) and (28). The classical force-free Taylor state corresponds to the line \( M = 0 \) in this figure; it becomes helically distorted for values of magnetic helicity larger than \( K = 4.08 \), which is in agreement with Ref. 28.

The interesting property of the helically distorted relaxed MHD states is their oscillatory nature. Indeed, substituting Eqs. (24), (25), (30), and (32) in Eqs. (1) and (2) with \( \epsilon = 0 \) one obtains that the phase \( \theta \) of the helical distortion is changing with time as

\[
\frac{\partial \theta}{\partial \tau} = -k \cdot (v_0 + \mu_2 b_0) = -m \left( \Omega + \frac{4\mu_2 CI_2(\lambda)}{\lambda} \right) - k_z (u + \mu_2),
\] (34)

where \( k = m/r e_x + k_e e_z \) is the wave-vector. In other words, the non-cylindrical part of the relaxed MHD state is the helical wave, which is propagating with a phase velocity \( v_{ph} = v_0 + \mu_2 b_0 \). At each point of the fluid, relaxed velocity and magnetic field are oscillating near the cylindrically symmetric state with the frequency \( \omega = \partial \theta / \partial \tau \) (Fig. 6).

Fig. 7, shows the \( F - \Theta \) diagram of the relaxed MHD states with different values of cross helicity \( M \) and angular momentum \( \Omega \). The reversal parameter \( F \) and pinch \( \Theta \) are defined here as

\[
F \equiv \frac{\langle B_x \rangle_{R=\infty} \Phi}{B_0} = 1 - CI_2(\lambda),
\] (35)

\[
\Theta \equiv \frac{\langle B_{\phi} \rangle_{R=\infty} \Phi}{B_0} = CI_1(\lambda),
\] (36)

where brackets \( \langle \rangle_{\Phi} \) denote averaging over \( \phi \) and \( z \). As follows from Fig. 7, the presence of the initial flow (non-zero cross helicity \( M \) or total angular momentum \( \Omega \)) in cylindrical plasma pinch affects the relaxed magnetic field significantly. This is due to the coupling of the velocity and the magnetic...
field that occurs in Eq. (22) through the term on the right hand side. Such coupling does not take place in the systems without axial symmetry where the angular momentum is not conserved, e.g., in a periodic rectangular box. In this case, a relaxed magnetic field corresponds to a force-free Taylor relaxed state [Eqs. (23)–(25)] with axial momentum \( u = 0 \) and angular momentum \( \Omega = 0 \).

### B. Energy minimization without momenta invariants

In order to obtain a relaxed state, which more realistically reflects the features of RFP, we have to exclude both axial \( I_2 \) and angular \( I_0 \) momenta invariants [Eqs. (16) and (17), respectively] from the energy minimization procedure for the following reasons. The angular momentum invariant \( I_0 \) is an artifact of the idealized geometry of periodic cylinder, it does not have an analogue in toroidal systems such as RFP. Although the axial momentum invariant \( I_2 \) has a counterpart in toroidal systems, we also discard it because it is not conserved in RFP experiment.\(^7\) Such “fragility” of the axial momentum invariant \( I_2 \) in real experiment can be explained by an important role of the viscous dissipation due to no-slip condition at the radial boundary. Results of this subsection reproduce in part results from Ref. 9 with corrections in Ref. 10.

The Euler equations describing a single-fluid MHD relaxed state under these assumptions are

\[
\begin{align*}
\sim \delta v : & \quad v_0 + \mu_2 b_0 = 0, \\
\sim \delta A : & \quad \nabla \times b_0 + 2\mu_1 b_0 + \mu_2 \nabla \times v_0 = 0, \\
\sim \delta \mu_\kappa : & \quad I_1 = \pi K, \quad I_2 = \pi M, \quad I_3 = \pi.
\end{align*}
\]

Equations (37) and (38) lead to one equation for a relaxed magnetic field,

\[
\nabla \times b_0 = \lambda b_0, \quad \lambda = \frac{2\mu_1}{\mu_2^2 - 1},
\]

which means that magnetic field relaxes to a force-free Taylor state. Notable feature of this relaxed state is that the flow is parallel to the magnetic field as seen from Eq. (37). The radial structure of this relaxed state is given by

\[
\begin{align*}
A_0 &= \frac{1}{2J_1(\lambda)} [J_1(\lambda) e_\phi + (J_0(\lambda) - J_0'(\lambda)) e_z] + \frac{C_0 h}{\lambda}, \\
b_0 &= \nabla \times A_0 = \frac{\lambda}{2J_1(\lambda)} [J_1(\lambda) e_\phi + J_0(\lambda) e_z] + C_0 h, \\
v_0 &= -\mu_2 b_0 - \frac{\mu_2 \lambda}{2J_1(\lambda)} [J_1(\lambda) e_\phi + J_0(\lambda) e_z] - \mu_2 C_0 h,
\end{align*}
\]

where helical part of the solution \( h \) is defined in Eq. (30). Here, the parameter \( \lambda \) (for cylindrically symmetric state) or amplitude of the helical part \( C_0 \) (for helically distorted state) have to be specified from the magnetic helicity constraint, \( l_1 = \pi K \), while Lagrange multiplier \( \mu_2 \) is determined from the cross helicity constraint, \( l_2 = \pi M \).

\[
\begin{align*}
K &= \frac{\lambda^2}{2} \left( 1 - \frac{J_0(\lambda) J_2(\lambda)}{J_1^2(\lambda)} \right) + \frac{D}{\lambda}, \\
W &= \frac{1}{2\pi l} \int b_0^2 \, dV = \frac{\lambda^2}{4} \left( 1 - \frac{J_0(\lambda) J_2(\lambda)}{J_1^2(\lambda)} \right) + \frac{\lambda J_0(\lambda)}{4J_1(\lambda)} + \frac{D}{2}, \\
\mu_2 &= -\frac{M}{2W},
\end{align*}
\]

where \( D \) is magnitude of the helical distortion given by Eq. (33) and \( W \) is volume-averaged magnetic energy of the relaxed state. The sum of kinetic and magnetic energy of the state is

\[
E = (1 + \mu_2^2) W = W + \frac{M^2}{4W}.
\]
distorted relaxed state occurs if value of magnetic helicity is larger than \( K = 4.08 \); the helical state in this case has the lowest possible value of \( \lambda = 3.11 \) corresponding to \( m = 1 \) and \( k_z = 1.23 \). This situation is violated when the cross helicity \( M \) is comparable to the magnetic energy \( W \) of the relaxed state. In fact, if \( M > 2W \) then the minimum energy state is cylindrically symmetric (Fig. 8).

In contrast to the relaxed states from Sec. III A, the states described by Eqs. (41)–(43) are always stationary and do not possess traveling helical waves.

IV. RELAXED STATES IN HALL MHD

In this section, we consider the Hall MHD (\( \epsilon \neq 0 \)) relaxed states corresponding to a minimum of energy [Eq. (7)] subject to constraints given by Eqs. (8), (11), and (14)–(17). As in Sec. III, we follow the standard procedure of minimization by applying the Lagrange multipliers method. Then the Euler equations of variational problem are

\[
\begin{align*}
\sim \delta \omega : & \quad \mathbf{v}_0 + \mu_2 (\mathbf{b}_0 + \epsilon \nabla \times \mathbf{v}_0) + \mu_4 \mathbf{e}_z + \mu_5 \mathbf{e}_\rho = 0, \\
\sim \delta \mathbf{A} : & \quad \nabla \times \mathbf{b}_0 + 2 \mu_1 \mathbf{b}_0 + \mu_2 \nabla \times \mathbf{v}_0 = 0, \\
\sim \delta \mu_k : & \quad I_1 = \pi K, \ I_2 = \pi \mu M, \ I_3 = \pi, \ I_4 = 2 \pi \Omega_b, \\
& \quad I_5 = \pi \mu u, \ I_6 = \frac{\pi \Omega}{2}.
\end{align*}
\]

Equations (48) and (49) can be reduced to one equation for magnetic field

\[
e \mu_2 \nabla \times \nabla \times \mathbf{b}_0 + \left( 1 - \mu_3^2 + 2 \epsilon \mu_1 \mu_2 \right) \nabla \times \mathbf{b}_0 + 2 \mu_1 \mathbf{b}_0 = 2 \mu_2 \mu_6 \mathbf{e}_z.
\]

A solution to this equation is the so-called double Beltrami flow,\textsuperscript{19}

\[\mathbf{b}_0 = C_1 \mathbf{b}_1 + C_2 \mathbf{b}_2 + \frac{\mu_2 \mu_6}{\mu_1} \mathbf{e}_z, \quad \nabla \times \mathbf{b}_{1,2} = \lambda_{1,2} \mathbf{b}_{1,2},\]

where \( \lambda_{1,2} \) are two roots of quadratic eigenvalue problem derived from homogeneous part of Eq. (51).

\[\lambda_{1,2} = \frac{-(1 - \mu_2^2 + 2 \epsilon \mu_1 \mu_2) \pm \sqrt{(1 - \mu_2^2 + 2 \epsilon \mu_1 \mu_2)^2 - 8 \epsilon \mu_1 \mu_2}}{2 \epsilon \mu_2} \tag{53}\]

The presence of two spatial scales associated with eigenvalues \( \lambda_1 \) and \( \lambda_2 \) is typical for the double Beltrami flows arising in the Hall MHD relaxation theory.\textsuperscript{14,19,22,23}

Similar to single-fluid MHD case, the most general solution to Eqs. (48) and (49) satisfying constraints \( I_3 = \pi, \ I_4 = 2 \pi \Omega_b, \ I_5 = \pi \mu u, \ I_6 = \pi \Omega/2 \) is a superposition of cylindrically symmetric and helical modes,

\[
\begin{align*}
\mathbf{A}_0 &= \frac{r}{2} e_\rho + \sum_{j=1,2} \left[ \frac{C_j}{\lambda_j} \left( J_1(\lambda_j) - J_1(\lambda_j) r \right) e_\rho + \frac{J_0(\lambda_j)}{\lambda_j} e_z + \mathbf{H}_j \right], \\
\mathbf{b}_0 &= \nabla \times \mathbf{A}_0 = e_z + \sum_{j=1,2} \left[ C_j (J_1(\lambda_j r) - 2 J_1(\lambda_j)) e_\rho + \lambda_j \right], \\
\mathbf{v}_0 &= \Omega e_\rho + \omega e_z - \sum_{j=1,2} \left( \lambda_j + 2 \mu_1 \right) \left[ \frac{4 \mu_2}{\lambda_j} \mu_6 \right] e_\rho + \frac{2 \mu_2}{\lambda_j} \right] e_z + \mathbf{H}_j, \tag{56}
\end{align*}
\]

where \( H_{1,2} \) are the helical parts of the solution, such that \( \nabla \times \mathbf{H}_{1,2} = \lambda_{1,2} \mathbf{H}_{1,2} \), and the coefficients \( C_{1,2} \) are

\[
\begin{align*}
C_1 &= \frac{\lambda_1 (\lambda_2 + 2 \mu_1) (\mu_1 + \mu_2 \Omega_b) J_3(\lambda_2) + \lambda_2 \lambda_1 \mu_2 \Omega_b J_3(\lambda_1)}{\lambda_1 (\lambda_2 + 2 \mu_1) J_1(\lambda_1) J_3(\lambda_2) + \lambda_2 \lambda_1 \mu_2 \Omega_b J_1(\lambda_1) J_3(\lambda_2)}, \\
C_2 &= \frac{\lambda_2 (\lambda_1 + 2 \mu_1) (\mu_1 + \mu_2 \Omega_b) J_3(\lambda_1) + \lambda_1 \lambda_2 \mu_2 \Omega_b J_3(\lambda_2)}{\lambda_1 (\lambda_2 + 2 \mu_1) J_1(\lambda_1) J_3(\lambda_2) + \lambda_2 \lambda_1 \mu_2 \Omega_b J_1(\lambda_1) J_3(\lambda_2)}.
\end{align*}
\]

Substituting these expressions into Eqs. (7), (8), and (11) we obtain the volume-averaged densities of the energy, the magnetic helicity and the generalized cross helicity,

\[
\begin{align*}
E \sim n l &= \frac{1}{2} \left[ u^2 + \frac{\Omega^2}{4} - 8 \frac{8 C_1 C_2 (\lambda_1 + 2 \mu_1) (\lambda_2 + 2 \mu_1) J_2(\lambda_1) J_2(\lambda_2)}{\lambda_1^2 \lambda_2^2 \mu_3^2} \\
&\quad + 2 C_1 C_2 \left( 1 + \frac{(\lambda_1 + 2 \mu_1) (\lambda_1 + 2 \mu_1)}{\lambda_1 \lambda_2 \mu_3^2} \right) \right] \\
&\quad \times \left( \frac{J_0(\lambda_2) J_1(\lambda_1) - J_0(\lambda_1) J_1(\lambda_2)}{\lambda_1 - \lambda_2} \right) \left( \frac{2 J_1(\lambda_1) J_2(\lambda_2)}{\lambda_1 \lambda_2} \right) \\
&\quad + \sum_{j=1,2} \left[ - \frac{4 C_1 J_2^2(\lambda_j) (\lambda_j + 2 \mu_1)^2}{\lambda_j^4 \mu_3^2} + \left( 1 + \frac{(\lambda_j + 2 \mu_1)^2}{\lambda_j^2 \mu_3^2} \right) \right] \\
&\quad \times \left( \frac{C_2^2}{2} (2 J_1(\lambda_j) J_2(\lambda_j) - J_2(\lambda_j)) + \frac{D_j}{2} \right), \tag{59}
\end{align*}
\]

FIG. 8. Domains occupied by cylindrically symmetric and helically distorted MHD relaxed states [Eqs. (41)–(43)] in the plane \( K - M \) (magnetic helicity–cross helicity). Grey areas correspond to helically distorted states with \( 3.11 < \lambda < 3.83 \).
\[ K = \frac{C_1 C_2}{\lambda_1 - \lambda_2} \left( J_1(\lambda_2) J_3(\lambda_1) - J_1(\lambda_1) J_3(\lambda_2) \right) \]
\[ + \sum_{j=1,2} \left[ \frac{2C_1 J_2(\lambda_j)}{\lambda_j} + \frac{2C_2^2}{\lambda_j} \left( J_2^2(\lambda_j) - J_1(\lambda_j) J_3(\lambda_j) \right) + \frac{D_j}{\lambda_j} \right], \tag{60} \]
\[ \dot{M} = u(1 + \epsilon \Omega) - \frac{C_1 C_2 \mu_1 (3 + \mu_2^2 - 2\epsilon \mu_1 \mu_2)}{\epsilon \mu_2^3} \]
\[ \times \left( \frac{J_0(\lambda_2) J_2(\lambda_2) - J_0(\lambda_1) J_2(\lambda_1)}{\lambda_1^2 - \lambda_2^2} - \frac{8J_2(\lambda_1) J_2(\lambda_2)}{\lambda_1^2 \lambda_2^2} \right) \]
\[ + \sum_{j=1,2} \left[ \frac{C_j (\lambda_j + \lambda_2 \mu_2^2 + 2\mu_1)}{\lambda_j^2 \mu_2^2} \left( \Omega J_3(\lambda_j) - w J_3(\lambda_j) \right) \right. \]
\[ + \frac{2u C_1 J_3(\lambda_j)}{\lambda_j} - \frac{J_2(\lambda_j) J_3(\lambda_j)}{\lambda_j} \right] \left. + \frac{D_j}{\lambda_j^2} \right]. \tag{61} \]

where \( D_{1,2} \) are magnitudes of the helical modes (note that helical eigenmodes of the curl operator corresponding to different eigenvalues \( \lambda_1 \neq \lambda_2 \) are orthogonal\textsuperscript{29},

\[ D_{1,2} = \frac{1}{\pi l} \int H_1^2 \, d^3 r. \]

Equations (60) and (61) with given \( K \) and \( M \) yield two conditions for determining two unknowns. For a cylindrically symmetric state with \( D_{1,2} = 0 \) these unknowns are Lagrange multipliers \( \mu_1 \) and \( \mu_2 \), or, equivalently, \( \lambda_1 \) and \( \lambda_2 \), since they are related to \( \mu_1 \) and \( \mu_2 \) by Eq. (53). For a single-helicity state with \( D_{1,2} \neq 0 \) and \( D_1 = 0 \), these unknowns are \( D_1 \) and \( D_2 \) since eigenvalue \( \lambda_1 \) of the helical mode is specified now by Eq. (31). For a double-helicity state, the unknowns are \( D_1 \) and \( D_2 \) since both eigenvalues \( \lambda_1 \) and \( \lambda_2 \) (and, therefore, \( \mu_1 \) and \( \mu_2 \)) are specified by Eq. (31).

The relaxed state found in such a way is not necessarily unique: for a given set of values of invariants there might be more than one solution corresponding to local minima of the energy functional. A true relaxed state should be selected as one of them with the lowest energy. One of the examples is illustrated in Fig. 9. It corresponds to the case when Eqs. (60) and (61) being solved for \( \mu_1 \) and \( \mu_2 \) yield multi-valued function \( \mu_2(K, M) \) with point of condensation at \( \mu_2 = 0 \). Because of non-uniqueness of this solution there exists a discrete set of energy levels corresponding to given values of the invariants \( K \) and \( M \). The lowest energy state in Fig. 9 corresponds to \( \mu_2 = 0 \) and is independent of the value of generalized cross helicity \( M \). The dependence of energy \( E \) on magnetic helicity \( K \) for such state is shown in Fig. 10. Note that the non-uniqueness of the cylindrically symmetric states is a characteristic feature of the Hall MHD model, it has no practically important applications in the single-fluid MHD model.

Despite the complexity of Eqs. (57)–(61) a simple analytical treatment of them is possible when \( \Omega_b = \Omega \), i.e., when boundary angular velocity defined by Eq. (15) is equal to averaged angular velocity defined by Eq. (17). In the rest of this section, we consider this case in more detail. The case \( \Omega_b \neq \Omega \) is considered in Appendix.

When \( \Omega_b = \Omega \) a state with the lowest energy corresponds to infinitely small \( \mu_2 \) (Fig. 9). In order to show this, we assume that \( \mu_2 \) is a small parameter and represent all quantities as a power series of \( \mu_2 \). Then

\[ \lambda_1 = -2\mu_1 - 2\mu_1 \mu_2^2 + o(\mu_2^2) \sim 1, \]
\[ \lambda_2 = -\frac{1}{\epsilon} \mu_2 - 2\mu_1 \mu_2^2 + o(\mu_2^2) \sim \mu_2^{-1}, \]

and

\[ C_1 = \frac{\lambda_1}{2J_1(\lambda_1)} + o(1) \sim 1, \]
\[ C_2 = -\frac{\lambda_1 J_1(\lambda_1) \mu_2^2}{2J_1(\lambda_1) J_3(\lambda_2)} + o(\mu_2) \sim \mu_2, \]

where it is also assumed that \( C_2 \sim \mu_2 \) (i.e., \( J_1(\lambda_2) \sim \mu_2 \)), this assumption is justified below [see Eq. (68)]. Taking into account leading order terms in Eqs. (54)–(56), we arrive at

\[ \frac{E_{\text{cr}}}{\epsilon K} \approx -C_{\text{hel}}. \]

\[ \frac{E}{\pi l} \approx K \]

\[ \mu_2 \approx 0. \]

\[ \lambda_2 \approx \mu_2. \]
asymptotically close to relaxed state: magnetic field varies slowly in radial direction. Equations (62)–(64) demonstrate the separation of scales in a relaxed state: magnetic field varies slowly in radial direction (with typical scale $1/\lambda_1$) while velocity has fast varying part (with typical scale $1/\lambda_2$).

In the limit $\mu_2 \to 0$, Eqs. (59)–(61) become

\begin{equation}
E \frac{u^2}{n} = \frac{\Omega^2}{4} + \frac{\lambda_2^2}{4} \left(1 - \frac{J_0(\lambda_2 J_1(\lambda_1))}{J_1(\lambda_1)^2} \right) + \frac{\lambda_2 J_0(\lambda_1)}{4 J_1(\lambda_1)} + \frac{D_1}{2},
\end{equation}

\begin{equation}
K = \frac{\lambda_1}{2} \left(1 - \frac{J_0(\lambda_1 J_1(\lambda_1))}{J_1(\lambda_1)^2} \right) + \frac{D_1}{\lambda_1},
\end{equation}

\begin{equation}
\dot{M} = uu (1 + \epsilon \Omega) + \frac{\Omega \lambda_2}{J_1(\lambda_1)} - \frac{2 \epsilon |\mu_2|^2 C_2}{\pi \mu_2^2}.
\end{equation}

As one can see from Eq. (65) the fast varying part of the velocity does not contribute into energy in leading order. The energy of the relaxed state given by Eqs. (62)–(64) is the sum of kinetic part, which is due to uniform axial flow $u$ and/or rigid azimuthal rotation $\Omega$, and magnetic part, which is due to the force-free magnetic field with fixed magnetic helicity $K$. It is easy to verify that Eq. (65) yields the lowest possible value of the energy under constrained magnetic helicity, axial and angular momenta. This validates the use of the limit $\mu_2 \to 0$ (or, equivalently, $\lambda_2 \to \infty$).

In order to complete description of the relaxed state, one has to use Eq. (66) to determine $\lambda_1$ for cylindrically symmetric state (with $D_1 = 0$) or $D_1$ for helically distorted state [with $\lambda_1$ satisfying Eq. (31)]. Amplitude $C_2$ is then determined from Eq. (67),

\begin{equation}
C_2^2 = \frac{\pi \mu_2^2}{2 \epsilon} \left|M - uu (1 + \epsilon \Omega) - \frac{\Omega \lambda_2}{J_1(\lambda_1)}\right| \sim \mu_2^2,
\end{equation}

which justifies assumption made above.

Though formally the terms in brackets in Eq. (64) are of order 1, they are negligible for any radius $r > 0$ in the limit $\lambda_2 \to \infty$. For large arguments $\lambda_2 r \gg 1$, Bessel functions entering Eq. (64) are bounded,

\begin{equation}
|J_m(\lambda_2 r)| \leq \frac{2}{\pi |\lambda_2| r} \to 0, \quad \lambda_2 \to \infty.
\end{equation}

Therefore, at any fixed radius $r > 0$ the relaxed velocity is asymptotically close to

\begin{equation}
v_0 = \Omega e_\phi + u e_r.
\end{equation}

We should also note that the boundary value of azimuthal velocity from Eq. (69) is

\begin{equation}
v_\phi \bigg|_{r=1} = \Omega,
\end{equation}

which is consistent with the assumed condition $\Omega_0 = \Omega$. So, Eqs. (62)–(64) describe the lowest energy Hall MHD state satisfying all imposed constraints given by Eqs. (8), (11), and (14)–(17) with $\Omega_0 = \Omega$.

Note that the relaxed magnetic field given by Eq. (63) is nothing else but a force-free Taylor state ($\nabla \times b_0 = \lambda_1 b_0$), i.e., it is a state with minimal magnetic energy under the constraints of constant magnetic helicity and total axial magnetic flux (Fig. 11). On the other hand, the asymptotic form of the relaxed velocity given by Eq. (69) is the flow, which minimizes kinetic energy with the constraints of constant axial and angular momenta. Thus, the Hall MHD relaxed state [Eqs. (62), (63), and (69)] can be found formally from the energy minimization procedure if one ignores the generalized cross helicity constraint.

This result is in agreement with the earlier considerations first made with the use of simple model in Ref. 20 and later for full Hall MHD model in Ref. 23. As mentioned there, it is a manifestation of the ill-posed variational problem: the constraint $I_2$ (generalized cross helicity) is more “fragile” than the target functional $E$ (energy), since $I_2$ contains higher-order derivatives of velocity $v$ in comparison with $E$. This result suggests that the relaxation in Hall MHD systems happens in such a manner as if there is no conservation of the generalized cross helicity. Such “fragility” of the generalized cross helicity in a real RFP experiment can be explained by its fast viscous decay due to an extra derivative of velocity and due to no-slip condition at the cylindrical boundary. The final relaxed state is trivial; it is the force-free magnetic field (Taylor state) and uniform axial flow and/or rigid rotation of plasma.

FIG. 11. Cylindrically symmetric Hall MHD relaxed state [Eqs. (54)–(56)] with the lowest energy, corresponding to a classical Taylor (force-free) state. The state is independent of generalized cross helicity $M$. Other parameters are: magnetic helicity $K = 2$, axial momentum $u = 0$, angular momentum $\Omega = 0$, boundary angular velocity $\Omega_0 = 0$, and Hall parameter $\epsilon = 0.1$. Note that such state has no flows since both momentum invariants $u$ and $\Omega$ are zero.
From the mathematical point of view, obtained Hall MHD relaxed state corresponds to a single-fluid MHD relaxed state [Eqs. (23)–(25)] with \( \mu_2 = 0 \). Similar to single-fluid MHD case, helically distorted Hall MHD relaxed state can be oscillatory in time if \( \psi_0 \neq 0 \). The non-cylindrical part of the relaxed magnetic field \( \mathbf{H}_l \) is the helical wave with the wave-vector \( \mathbf{k} = m/r_{\psi} + k_z e_z \) and the phase velocity \( v_{ph} = \psi_0 \), so the frequency of oscillations is [cf. Eq. (34)],

\[
\omega = -k \cdot \mathbf{v}_0 = -m\Omega - k_z u. \tag{70}
\]

The transition of the force-free magnetic field to helically distorted state with \( \lambda = 3.11 \) occurs when magnetic helicity \( K > 4.08 \) [Sec. III A].

We stress here that the presence of the flow in the Hall MHD relaxed states is fully due to the inclusion of momenta invariants \( I_s \) and \( I_s \) into energy minimization procedure. If we ignore them to better adapt the model to the features of RFP experiments [see explanation in Sec. III B], the final Hall MHD state will be just force-free (Taylor) magnetic field with no plasma flows. Obviously, such state cannot explain the flows observed in RFP experiments.

V. COMPARISON WITH EXPERIMENT

In this section, we compare theoretically predicted relaxed velocity with the plasma velocity measurements in the Madison symmetric torus (MST) RFP experiment taken from Ref. 7. For comparison we use single-fluid MHD relaxed state [Eqs. (41)–(43)] from Sec. III B (with flow parallel to the force-free magnetic field), since it is obtained under assumptions consistent with realistic RFP experiment. As pointed out in Sec. IV, Hall MHD relaxed state under these assumptions does not have flows, so it cannot be used for comparison.

The MST RFP has a major radius \( R = 1.5 \) m and a minor radius \( a = 0.5 \) m; the other plasma parameters relevant to our study are as follows: the line-averaged density is \( n \approx 1 \times 10^{13} \text{ cm}^{-3} \), the plasma current is \( I_p = 200 - 250 \) kA, the reversal parameter is \( F \approx -0.2 \), and the pinch parameter is \( \Theta \approx 1.7 \).

MST plasmas exhibit quasiperiodic (sawtooth) oscillations, which are characterized by sudden bursts in all plasma diagnostics. Sawtooth oscillations in the MST consist of a fast crash phase (~0.1 ms) and a slow recovery phase (~3 ms). During the crash plasma relaxes towards its minimum energy state.\(^3\)\(^4\) This relaxation event is accompanied by a rapid flattening of the plasma parallel momentum profile (Fig. 12).

In order to compare predictions of the relaxation theory with experimental observations presented in Fig. 12, we find parallel momentum normalized by the magnetic field using Eqs. (42) and (43),

\[
q_{||} = \frac{\mathbf{v}_0 \cdot \mathbf{b}_0}{\mathbf{b}_0} = -\mu_2, \tag{71}
\]

where \( \mu_2 \) is constant independent of radius and can be calculated from Eq. (46). This result is in agreement with the observed flattening of the parallel momentum profile during relaxation events. The momentum profile flattening is mostly noticeable in the plasma core, while the changes at the edge are small (Fig. 12). This is similar to the incomplete relaxation of the parallel current profile observed in these relaxation events.\(^3\)

We emphasize that the relaxation theory is developed for isolated systems, while the MST experiment is an open system with external energy supply. However, even in open systems, where energy relaxation happens much faster than dissipative decay of the invariants, the principle of selective decay remains valid, and the present theory can be applied. This is the case of the MST RFP experiment, in which the characteristic energy relaxation time (crash phase, \( t_{\text{crash}} \approx 0.1 \text{ ms} \)) is much shorter than the dissipation time (\( t_{\text{diss}} \approx 100 \text{ ms} \)).

VI. CONCLUSION

In the present paper, the application of the Taylor relaxation theory to a cylindrical plasma pinch was generalized by inclusion into energy minimization procedure of velocity related ideal invariants. We obtained the minimum energy (relaxed) states within the framework of both single-fluid and Hall MHD. When we began the calculations, our motivation and expectation was that more general Hall MHD would lead to more diverse relaxed states than single-fluid MHD, and that these might help to explain the rotation that seems to be ubiquitous in the RFP experimental results. To our surprise, it was not so. It turned out that accurate minimization of the energy \( E \) in the Hall MHD model leads only to the force-free Taylor magnetic field and simple flows (rigid rotation and/or uniform axial flow), while MHD relaxed states are much more complex and diverse. Physically this suggests that the difference between two models appears in the process of relaxation in the way that makes Hall MHD states simpler. From the mathematical point of view, this is due to ill-posedness of the variational problem in the Hall MHD, which is related to “fragility” of the generalized cross helicity \( I_2 \) explained at the end of Sec. IV. Our formal analysis shows that accurate energy minimizations in the Hall MHD with or without generalized cross helicity lead to the
same result. Therefore, during Hall MHD relaxation in a weakly dissipative system one should expect rapid viscous decay of the generalized cross helicity—it is not a robust invariant and does not affect the final relaxed state.

In both single-fluid and Hall MHD models, depending on initial values of the corresponding invariants the relaxed states can be either cylindrically or helically symmetric. The interesting property of the helically symmetric relaxed states is their oscillatory behavior. The helical distortion of relaxed velocity and magnetic field acts as a helical wave, propagating on a cylindrically symmetric background. The phase velocity of such a wave is not zero only when the initial values of velocity related invariants are not zero.

Like all variational theories of plasma relaxation, the present calculation is silent as to the details of the dynamics that are responsible for the relaxation process. The only requirement is that they preserve the robust invariants assumed during the variational procedure. The comparison of the theoretically predicted relaxed states with MST RFP experiment (Sec. V) shows that the ideal single-fluid MHD results are in better qualitative agreement with the experimental data than the Hall MHD results. The single-fluid MHD yields more complex states with flow and agrees better with experimental results, than Hall MHD. The Hall effect (or other non-MHD effects) can influence the relaxation dynamics, but the final relaxed state in the experiment resembles the minimum energy state from the single-fluid MHD. Further insight in this regard requires more detailed experimental measurements and large scale computer simulations.

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APPENDIX: RELAXED STATES IN HALL MHD FOR \( \Omega_b \neq \Omega \)

In general case, when \( \Omega_b \neq \Omega \), there are no simple analytical solutions to Eqs. (57)–(61). In order to find, for example, a cylindrically symmetric relaxed state (so, \( D_{1,2} = 0 \)), one has to solve Eqs. (60) and (61) for \( \mu_1 \) and \( \mu_2 \) (or, equivalently, for \( \lambda_1 \) and \( \lambda_2 \)), substitute found values into Eqs. (57)–(59), and if there are several solutions, choose the state with the lowest energy. Such procedure can be realized numerically and it leads to non-trivial profiles of magnetic field and velocity (Fig. 13). It should be emphasized here that the non-trivial relaxed states obtained in this way do not always have a smooth transition to a trivial state [Eqs. (62), (63), and (69)] when \( \Omega_b \rightarrow \Omega \).

We note that the Lagrange multipliers method applied to a minimization problem with constraints gives a solution from a class of continuous, differentiable functions. If we are not restricted to this class of functions, it is possible to construct a discontinuous relaxed state, which has lower energy than any other state. Indeed, let us represent velocity in the form,

\[
v = \mathbf{u} + (\Omega_0 - \Omega)b(r)\mathbf{e}_\phi, \quad (A1)
\]

where \( \mathbf{u} \) is assumed to be spatially continuous and \( b(r) \) is a discontinuous function describing a spike at the boundary,

\[
S(r) = \begin{cases} 
0, & r \neq 1; \\
1, & r = 1.
\end{cases}
\]

For such representation of velocity, the generalized cross helicity constraint now has a form,

\[
\hat{I}_2 = \int \mathbf{u} \cdot (\mathbf{b} + \frac{\epsilon}{2} \nabla \times \mathbf{u}) d^3 \mathbf{r} + \pi i \epsilon (\Omega_0 - \Omega) u_\phi \bigg|_{r=1},
\]

\[
(A2)
\]

and the azimuthal component of \( \mathbf{u} \) should satisfy boundary condition [cf. Eq. (15)],

\[
\int_0^{2\pi} u_\phi \bigg|_{r=1} d\varphi = 2\pi \Omega.
\]

(A3)

In terms of continuous differentiable functions \( \mathbf{b} \) and \( \mathbf{u} \), the problem is now reduced to the case \( \Omega_b = \Omega \) with modified generalized cross helicity [Eq. (A2)]. Hence, continuous part of the relaxed state is determined by Eqs. (62), (63), and (69). The spike discontinuity at the boundary introduced in Eq. (A1) enters the azimuthal component of the relaxed velocity,

\[
v_0 = \Omega \epsilon \mathbf{e}_\phi + \epsilon \mathbf{e}_z + (\Omega_0 - \Omega) S(r)\mathbf{e}_\phi.
\]

(A4)

The state given by Eqs. (62), (63), and (A4) has the lowest possible energy under imposed constraints. Therefore, isolated dissipative Hall MHD system will tend to relax toward this state.

A discontinuity at the boundary in the relaxed azimuthal velocity Eq. (A4) is due to the assumption that axial flux of the fluid vorticity \( I_4 \) [Eq. (15)] is conserved. However, similar to the generalized cross helicity, this constraint is more "fragile" than the energy since it contains higher-order derivatives of the velocity. As a result, ideal invariant \( I_4 \) is more...
susceptible to dissipation than energy and cannot be considered as a constant during relaxation. Ignoring this invariant in the energy minimization procedure, we obtain Eq. (69) for the relaxed velocity without any discontinuities.

The relaxed state obtained in such way is exactly the state described by Eqs. (62), (63), and (69), and all results of Sec. IV also apply. So, the final relaxed state of the cylindrical pinch in the Hall MHD is always a trivial state with the force-free magnetic field (Taylor state) and uniform axial flow and/or rigid rotation of plasma.