Finite electron temperature effects on interferometric and polarimetric measurements in fusion plasmas

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Finite electron temperature effects on interferometry and polarimetry measurements for burning plasma are considered with particular focus on analytically understanding the role of weakly relativistic effects. Development of a new iterative technique, in the limit when the probing wave frequency is much higher than the electron cyclotron frequency, yields the dispersion relation to lowest (linear) order in $T_e/m_ec^2 \ll 1$. Perturbative treatment of the wave phase and polarization is presented in a form suitable for interpretation of experimental data. Previous analysis of the problem included nonrelativistic calculations only. Herein, it is shown that relativistic effects are equally important. Theoretical results are in agreement with computations and can be used for benchmarking of ray tracing codes. The implication of finite temperature effects on future burning plasma interferometer diagnostics is discussed. © *2007 American Institute of Physics*.
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I. INTRODUCTION

The refractive indices and evolution of polarization for high-frequency electromagnetic waves are of major interest for density and magnetic field diagnostics in laboratory and astrophysical plasmas. The cold plasma dispersion relation has been widely used for many years to interpret interferometry and polarimetry measurements. Relativistic kinetic theory of electromagnetic waves and the covariant ray tracing formalism were intensively studied for astrophysical applications (see, for example, Refs. 1 and 2). Relativistic effects on reflectometry in laboratory plasmas (refractive indices, cutoffs) were treated numerically in Ref. 3 on the basis of a computationally convenient expression for the weakly relativistic dielectric tensor. Recently, the effects of thermal electron motion on interferometry and polarimetry analysis were analytically investigated and found to be potentially measurable[4] in the electron temperature range typical for JET,[5] ITER,[6] and other high-temperature tokamaks. The lowest-order corrections in $\tau\,(=T_e/m_ec^2 \ll 1)$ to interferometric effects (including Faraday rotation and Cotton-Mouton effects) were calculated in Ref. 4 on the basis of the nonrelativistic dielectric tensor ϵ_{ij} for magnetized plasmas. The nonrelativistic approach was justified by arguing that at wave frequencies ω significantly higher than the electron gyrofrequency ω_{ce}, relativistic effects can be ignored. We revisit this problem by analyzing the relativistic kinetic equation for a magnetized plasma, and show[7] that thermal corrections caused by the weakly relativistic effects are important for $\omega \gg \omega_{ce}$. They are also linear with τ, being comparable to the nonrelativistic thermal corrections but with opposite sign. The importance of relativistic effects for precise treatment of interferometry and polarimetry measurements at finite electron temperature motivated our interest in developing a more complete theoretical analysis of the problem.

The paper is organized as follows. Section II gives a description of a new analytical approach based on the iterative scheme of the solution of the kinetic equation with the use of the small parameter $\omega_{ce}/\omega \ll 1$. The quantitative analysis is preceded by a qualitative introduction of two different types of the thermal effects: the dispersive nonrelativistic corrections caused by the Doppler effect and weakly relativistic nondispersive contributions caused by relativistic mass dependence on the velocity. The linear electron temperature corrections for the dielectric tensor are presented in the Stix reference frame with the z axis oriented along the equilibrium magnetic field. In Sec. III, the polarization equation is analyzed in the wave coordinate system with the z axis oriented along the laser beam. Perturbative treatment of the Jones matrix yields the thermal corrections to the refractive indices and eigenvectors of the normal waves. The evolution equation for the Stokes vector is derived with the relativistic effects taken into account. A brief discussion and conclusions are drawn in Sec. IV. Details of the calculations are presented in Appendixes A and B.

II. HIGH-FREQUENCY DIELECTRIC TENSOR IN WARM PLASMA

A. Qualitative consideration

For a transverse electromagnetic wave, thermal corrections to the refractive index are caused by the Doppler shift of the frequency $\omega-\mathbf{k}\cdot\mathbf{v}$, where $\mathbf{k}$ and $\mathbf{v}$ are the wave vector and the electron thermal velocity. For the longitudinal plasma wave, in addition to the Doppler shift there is a contribution from the electron pressure perturbation. In the transverse case, the electron pressure is not perturbed, but due to the Doppler effect the magnitude of electron oscillatory velocity is sensitive to the electron thermal velocity. Consider, for example, two electrons moving with thermal velocities $\pm v$ in the direction of the wave vector. Their contribution to the transverse current induced in plasma by the wave leads to a correction proportional to v^2,

$$\mathbf{j}=\mathbf{j}_+ + \mathbf{j}_- = \frac{ie^2}{m_e\gamma}\left(\frac{1}{\omega-kv}+\frac{1}{\omega+kv}\right)\mathbf{E} \simeq \frac{2ie^2}{m_e\gamma\omega}\left(1+\frac{k^2v^2}{\omega^2}\right)\mathbf{E}, \quad (1)$$

where m_e is the electron rest mass and relativistic factor $\gamma=(1-v^2/c^2)^{-1/2}$. Averaging the k^2v^2/ω^2 term over the equilibrium Maxwellian distribution yields a function of electron temperature. The lowest-order term in expansion of this function at $\tau\ll 1$ is linear in T_e. Correspondingly, one can ignore relativistic corrections by letting $\gamma=1$ and using the nonrelativistic Maxwellian distribution function for averaging. Due to the nonrelativistic nature and the presence of the k^2 factor, we will refer to this linear T_e contribution as the nonrelativistic dispersive correction. Dispersive corrections lead to a transverse current increase with respect to the case of cold electrons and, thus, result in a decrease of the refractive index $N=kc/\omega$ by a factor $\propto\tau N^2$. Since the phase velocity of a high-frequency electromagnetic wave is close to the speed of light ($N^2\simeq 1$), the nonrelativistic thermal corrections are of order τ.

The purpose of this paper is to point out that in addition to the above nonrelativistic dispersive corrections, there are relativistic corrections linear in τ that were not previously addressed in analytic treatments of the interferometric effects. They originate from the first proportional to unity term in Eq. (1) and are caused by the increase of the electron mass $m_e\gamma$ due to the dependence of γ on the electron thermal velocity. Averaging over the electron distribution function leads, effectively, to an electron mass increase with the electron temperature. In the weakly relativistic limit, the corresponding corrections are proportional to τ ($\gamma-1\simeq T_e/m_ec^2$) and, thus, are of the same order as the dispersive ones. Since they are not proportional to k^2 or N^2, we will refer to them as the nondispersive relativistic thermal corrections. Obviously, the larger electron mass slows the electron response, thereby effectively increasing the refractive indices. As a result, the sign of this contribution is opposite to the dispersive correction. To distinguish the above two mechanisms, we will use the terms "nonrelativistic" or, equivalently, "dispersive" for the former type of the corrections, and "relativistic" or "nondispersive" for the latter one. Both effects must be included simultaneously for accurate interpretation of the interferometric measurements.

B. Relativistic kinetic equation for magnetized plasma

Due to the short wavelength of the electromagnetic waves used for interferometric diagnostics, their typical frequency ω greatly exceeds the characteristic plasma frequencies such that

$$\omega \gg \omega_{pe} \simeq \omega_{ce} \gg \omega_{ci} \gg \nu_{ei}. \quad (2)$$

Under these conditions, the main contribution to plasma linear response is given by the electrons while the ion motion can be ignored. The electron response is treated on the basis of the relativistic Vlasov equation for electron distribution function $F(\mathbf{r},\mathbf{p},t)$ in uniform magnetic field $\mathbf{B}_0$, which is perturbed by the fast oscillating magnetic and electric field $\mathbf{E}$ of the wave. In linear approximation with respect to the small electric field, the distribution function $F(\mathbf{r},\mathbf{p},t)$ is divided into a stationary equilibrium part $f(\mathbf{p})$ and a perturbation $\delta f(\mathbf{p},r,t)$,

$$F(\mathbf{r},\mathbf{p},t)=f(\mathbf{p})+\delta f(\mathbf{p},r,t). \quad (3)$$

Presenting $\delta f(\mathbf{p},r,t)$ as a sum of the Fourier harmonics $\propto\exp i(\mathbf{k}\cdot\mathbf{r}-\omega t)$ yields a nonhomogeneous first-order ordinary differential equation for the Fourier components of δf. Using a spherical reference frame (p,θ,ϕ) with the z axis parallel to the unperturbed magnetic field $\mathbf{B}_0$ and assuming that the unperturbed distribution function is isotropic, $f(\mathbf{p})=f(p)$, gives

$$-i\left(\omega-\frac{\mathbf{k}\cdot\mathbf{p}}{m_e\gamma}\right)\delta f+\frac{\omega_{ce}}{\gamma}\frac{\partial\delta f}{\partial\phi}=-\frac{e\mathbf{E}\cdot\mathbf{p}}{p}\frac{\partial f}{\partial p}, \quad \omega_{ce}=\frac{|e|B_0}{m_ec}, \quad (4)$$

where the relativistic factor γ describes the relationship between particle momentum and velocity, $\mathbf{p}=m\mathbf{v}\gamma$. The same notation $\delta f(\mathbf{p})$ is used for the Fourier harmonics of $\delta f(\mathbf{p},r,t)$. The factor γ is also a measure of the relativistic mass increase caused by electron thermal motion and gives rise to the relativistic corrections to the plasma dielectric tensor.

Linear response is determined by the currents induced in plasma,

$$\mathbf{j}=\frac{n_0e}{m_e}\int_0^\infty\left(\frac{\mathbf{p}}{\gamma}\right)p^2dp\int_0^\pi \sin\theta d\theta\int_0^{2\pi}d\phi\,\delta f(p,\theta,\phi), \quad (5)$$

where n_0 is the equilibrium plasma density, and the equilibrium relativistic Maxwellian distribution function $f(p)$ is assumed to be normalized to unity,

$$4\pi\int_0^\infty f(p)p^2dp=1. \quad (6)$$

Using the definition of the displacement vector $\mathbf{D}$ and expressing $\mathbf{j}$ as a function of the electric field yields the elements of the dielectric tensor ϵ_{ij},

$$\mathbf{D}=\mathbf{E}+\frac{4\pi i}{\omega}\mathbf{j}, \quad D_i=\epsilon_{ij}E_j. \tag{7}$$

The standard calculation method is based on the exact integration of Eq. (4). The constant of integration is determined by the periodicity of δf on azimuthal angle ϕ (see, for example, Ref. 8),

$$\delta f=-\frac{e\gamma}{\omega_{ce}}\int_{\infty}^{\phi}d\phi'\left(\mathbf{E}\cdot\frac{\partial f}{\partial \mathbf{p}}\right)\exp\left[\frac{i\gamma}{\omega_{ce}}\int_{\phi}^{\phi'}\left(\omega-\frac{\mathbf{k}\cdot\mathbf{p}}{m\gamma}\right)d\phi''\right]. \tag{8}$$

Expanding the exponential function in series of the Bessel functions summed over the harmonics of ϕ and performing angular integrations (5) yields the elements of ϵ_{ij} that have, in the relativistic case, a well known form (see, for example, Ref. 9),

$$\epsilon_{ij}\propto\sum_n\int\frac{J_n^2(k_\perp p_\perp/\omega_{ce}m)}{\omega\gamma(p)-n\omega_{ce}-k_\parallel p_\parallel/m}\frac{\partial f}{\partial p}dp_\parallel p_\perp dp_\perp. \tag{9}$$

These expressions are then expanded over the small parameter τ. Because of the infinite series, resonant factors in the denominator, and relatively large value of the argument of the Bessel functions, this presentation is not suitable for the expansion in powers of τ.

A more tractable form known as the weakly relativistic dielectric tensor was suggested in Ref. 10. It was derived from the original Trubnikov results[11] and had a form of the double series expansion in powers of the factor $(k_\perp v_{Te}/\omega_{ce})^2$. Since the coefficients of this expansion depend on T_e and have a complicated matrix form, this presentation is also difficult for analytical treatment.

Instead of using these general expressions, we have developed a simple calculation scheme adequate for the case of practical interest of a high-frequency electromagnetic wave with $\omega \gg \omega_{ce}$. This allows us to avoid exact integration of Eq. (4) and to find corrections linear in T_e by means of successive differentiations of simple trigonometric functions. The approach is based on the recursion equation (A7). From the point of view of the exact solution (8), it is equivalent to the expansion in powers of ω_{ce}/ω by means of successive integrations by parts with the use of the relationship

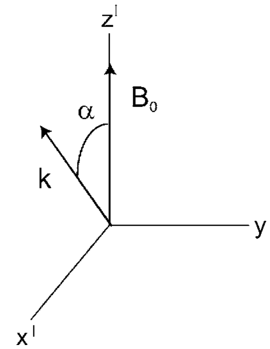

FIG. 1. The Stix reference frame $x'y'z'$ with $\mathbf{z}'\parallel\mathbf{B}_0$ and $\mathbf{k}$ in the x',z' plane.

$$d\phi'\exp\left(\frac{i\gamma\omega\phi'}{\omega_{ce}}\right)=\frac{\omega_{ce}}{i\gamma\omega}d\left[\exp\left(\frac{i\gamma\omega\phi'}{\omega_{ce}}\right)\right]. \tag{10}$$

The calculation details are described in Appendix A. First, we use the Stix reference frame x',y',z' with the z' axis oriented along $\mathbf{B}_0$ and the $\mathbf{k}$ vector in the x',z' plane ($k_{x'}=k\sin\alpha$, $k_{y'}=0$, $k_{z'}=k\cos\alpha$) (see Fig. 1). The dielectric tensor is presented as a superposition $\epsilon'_{ij}=\epsilon'^{(c)}_{ij}+\Delta\epsilon'^{(T)}_{ij}$, including a cold plasma part $\epsilon'^{(c)}_{ij}$ and the first-order temperature correction $\Delta\epsilon'^{(T)}_{ij}$ proportional to τ. The thermal part is further divided in two physically different parts, $\Delta\epsilon'^{(T)}_{ij}=\Delta\epsilon'^{(ND)}_{ij}+\Delta\epsilon'^{(D)}_{ij}$, where $\Delta\epsilon'^{(ND)}_{ij}$ describes weakly relativistic nondispersive corrections while the term $\Delta\epsilon'^{(D)}_{ij}$ is responsible for nonrelativistic dispersive effects. Calculations show (see Appendix A) that these terms can be written as

$$\epsilon'^{(c)}_{ij}=\delta_{ij}-\frac{X}{1-Y^2}\begin{pmatrix}1 & -iY & 0\\ iY & 1 & 0\\ 0 & 0 & 1-Y^2\end{pmatrix}, \quad X=\frac{\omega_{pe}^2}{\omega^2}, \quad Y=\frac{\omega_{ce}}{\omega}, \quad \omega_{pe}^2=\frac{4\pi n_0e^2}{m_e}, \tag{11}$$

$$\Delta\epsilon'^{(ND)}_{ij}=\frac{5\tau X}{2(1-Y^2)^2}\begin{pmatrix}1+Y^2 & -2iY & 0\\ 2iY & 1+Y^2 & 0\\ 0 & 0 & (1-Y^2)^2\end{pmatrix}, \tag{12}$$

$$\begin{aligned}\Delta\epsilon'^{(D)}_{ij}=\tau XN^2&\begin{pmatrix}-(1+2\sin^2\alpha) & 3iY(1+\sin^2\alpha) & -\sin 2\alpha\\ -3iY(1+\sin^2\alpha) & -1 & -(3/2)iY\sin 2\alpha\\ -\sin 2\alpha & (3/2)iY\sin 2\alpha & -(1+2\cos^2\alpha)\end{pmatrix}\\ &+\tau XN^2Y^2\begin{pmatrix}-6-9\sin^2\alpha & 0 & -2\sin 2\alpha\\ 0 & -6-7\sin^2\alpha & 0\\ -2\sin 2\alpha & 0 & -\sin^2\alpha\end{pmatrix}, \quad N^2=k^2c^2/\omega^2.\end{aligned} \tag{13}$$

Since the plasma is treated as being nondissipative, all three parts are Hermitian, $\epsilon'_{ij}=\epsilon'^{*}_{ji}$. The second term in Eq. (13), proportional to Y^2, is needed for correct treatment of the Cotton-Mouton effect. Note that this term was not presented in the similar equation (14) in Ref. 4. However, the final evolution equation for the Stokes vector is calculated in Ref. 4 with this term taken into account. Keeping this caveat (disagreement of the intermediate results) in mind, our results for the dispersive part (13) are consistent with Ref. 4.

The key finding of our paper is an additional nondispersive thermal correction (12) that does not depend on N^2 and originates from the relativistic effects caused by the factor $\gamma(p)$. Since the refractive index of high-frequency electromagnetic waves has $N^2 \simeq 1$, the nonrelativistic dispersive and weakly relativistic nondispersive corrections are of the same order of magnitude. Moreover, comparing their absolute values shows that for the most important elements $\epsilon'_{xx}, \epsilon'_{xy}, \epsilon'_{yy}$, the weakly relativistic nondispersive corrections are larger than the dispersive ones and have the opposite sign.

The correctness of weakly relativistic nondispersive contribution (12) is confirmed by the comparison with Ref. 12 devoted to a relativistic dielectric tensor in plasma without magnetic field. The elements of ϵ_{ij} are calculated in Ref. 12 in the reference frame with the z' axis parallel to $\mathbf{k}$. In order to compare Eqs. (11)–(13) with Ref. 12, we set $\alpha=0$ to match the reference frames and $Y=0$ for limiting transition to zero magnetic field. This yields the dielectric tensor (11)–(13) in the form

$$\epsilon_{ij} = \delta_{ij}\left[1 - X + \tau X\left(\frac{5}{2} - N^2\right)\right] - 2\tau X \frac{k_i k_j c^2}{\omega^2} \tag{14}$$

that coincides with the weakly relativistic limit of the functions ϵ_L and ϵ_T calculated in Ref. 12.

The main argument of Ref. 4 in favor of ignoring the relativistic effects was that they could be important in series expansions (9) only at large $n \simeq \omega/\omega_{ce}$. Since the contribution from large n is small, relativistic effects were ignored and the nonrelativistic dielectric tensor was assumed to give an accurate estimate of linear in T_e corrections. If this were so, the limiting transition $B_0 \to 0$, which is consistent with the large $n \simeq \omega/\omega_{ce} \to \infty$ assumption, would result in the expressions containing no terms of the relativistic origination. This contradicts Eq. (14), where the weakly relativistic factor 5/2 is explicitly presented. Thus, both the calculations[12] and our result (12) confirm that the relativistic effects are important and yield contributions comparable with nonrelativistic terms. This indicates that accurate analysis of linear corrections in T_e requires the use of the relativistic Vlasov kinetic equation.

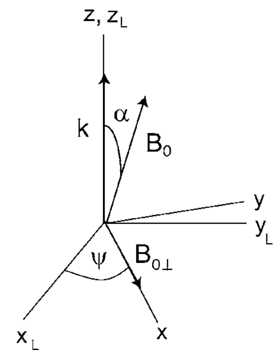

FIG. 2. Local wave reference frame x,y,z with $\mathbf{z}\|\mathbf{k}$ and the x axis along $\mathbf{B}_{0\perp}$ and the laboratory reference frame x_L, y_L, z_L with $\mathbf{z}_L\|\mathbf{k}$ and x_L and y_L axes fixed with respect to the experimental device. The spatially varying azimuth $0 \leqslant \psi \leqslant \pi$ is the angle in the x_L, y_L plane between Ox_L and the direction of $\mathbf{B}_{0\perp}$.

For analysis of the wave polarization, it is convenient to transfer the dielectric tensor (11)–(13) to a new reference frame x,y,z. The z axis is oriented along the vector $\mathbf{k}$ while the vector of the unperturbed magnetic field is in the x,y plane and has the Cartesian coordinates $\mathbf{B}_0 = (B_0 \sin\alpha, 0, B_0 \cos\alpha)$ (see Fig. 2). The transformation is achieved by two successive rotations of the initial reference frame around the y' axis by the angle α and around the z' axis by the angle π. The new tensor ϵ_{ij} is related to the initial one ϵ'_{ij} as follows:

$$\epsilon = \mathbf{T}\cdot\epsilon'\cdot\mathbf{T}^{-1}, \quad \mathbf{T} = \begin{pmatrix} -\cos\alpha & 0 & \sin\alpha \\ 0 & -1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{pmatrix}, \tag{15}$$

where $\mathbf{T}$ is the transformation matrix. Applying this transformation for the dielectric tensors (11)–(13) yields three parts,

$$\epsilon_{ij}^{(c)} = \delta_{ij} - \frac{X}{1-Y^2}\begin{pmatrix} 1 - Y^2\sin^2\alpha & -iY\cos\alpha & -Y^2\sin\alpha\cos\alpha \\ iY\cos\alpha & 1 & -iY\sin\alpha \\ -Y^2\sin\alpha\cos\alpha & iY\sin\alpha & 1 - Y^2\cos^2\alpha \end{pmatrix}, \tag{16}$$

$$\Delta\epsilon_{ij}^{(ND)} = \frac{5}{2}\tau X C\begin{pmatrix} 1 - A\sin^2\alpha & -iB\cos\alpha & -A\cos\alpha\sin\alpha \\ iB\cos\alpha & 1 & -iB\sin\alpha \\ -A\cos\alpha\sin\alpha & iB\sin\alpha & 1 - A\cos^2\alpha \end{pmatrix}, \tag{17}$$

$$\Delta\epsilon_{ij}^{(D)} = -\tau X N^2\begin{pmatrix} 1 + Y^2(6 - 5\sin^2\alpha) & -3iY\cos\alpha & 5Y^2\sin 2\alpha \\ 3iY\cos\alpha & 1 + Y^2(6 + 7\sin^2\alpha) & -6iY\sin\alpha \\ 5Y^2\sin 2\alpha & 6iY\sin\alpha & 3 + 15Y^2\sin^2\alpha \end{pmatrix}, \tag{18}$$

where

$$A=\frac{Y^2(3-Y^2)}{1+Y^2},\quad B=\frac{2Y}{1+Y^2},\quad C=\frac{(1+Y^2)}{(1-Y^2)^2}. \tag{19}$$

The dispersive part $\Delta\epsilon_{ij}^{(D)}$ is obtained by the power expansion in Y up to Y^2 order. In contrast to this, the nondispersive part (17) is found exactly and is valid at any Y. To calculate Eq. (17) with the same Y^2 accuracy, the constants (19) are simplified by power expansion in Y,

$$A\simeq 3Y^2,\quad B\simeq 2Y,\quad C\simeq 1+3Y^2. \tag{20}$$

This yields the final form of (17),

$$\Delta\epsilon_{ij}^{(ND)}=\frac{5}{2}\tau X\begin{pmatrix} 1+3Y^2\cos^2\alpha & -2iY\cos\alpha & -3Y^2\cos\alpha\sin\alpha \\ 2iY\cos\alpha & 1+3Y^2 & -2iY\sin\alpha \\ -3Y^2\cos\alpha\sin\alpha & 2iY\sin\alpha & 1+3Y^2\sin^2\alpha \end{pmatrix}. \tag{21}$$

III. EFFECT OF ELECTRON THERMAL MOTION ON THE EVOLUTION OF POLARIZATION

The electric field of the wave is determined by the real part of the expression

$$\Re e\{\mathbf{E}\exp(ikz-i\omega t)\}, \tag{22}$$

where $\mathbf{E}$ is a constant complex vector. Slow spatial variations of $\mathbf{E}$ can be considered in the higher-order approximation within the scope of the geometrical optics. The homogeneous system of the Maxwell equations for three components of $\mathbf{E}$,

$$(\delta_{ij}N^2-n_in_jN^2-\epsilon_{ij})E_j=0,\quad \mathbf{n}=\mathbf{k}/k,\quad i,j=1,2,3, \tag{23}$$

determines the dispersion and the polarization properties of the wave. Expressing E_z in terms of E_x and E_y from the z component of Eq. (23) and substituting into the x and y components yields two coupled equations for E_x and E_y that are suitable for the analysis of polarization (Jones equations),

$$\begin{pmatrix} N^2-\eta_{xx} & -\eta_{xy} \\ -\eta_{yx} & N^2-\eta_{yy}\end{pmatrix}\begin{pmatrix}E_x\\E_y\end{pmatrix}=0,$$
$$\eta_{ij}=\epsilon_{ij}-\epsilon_{iz}\epsilon_{zj}/\epsilon_{zz},\quad i,j=1,2. \tag{24}$$

According to (24), the Jones tensor η_{ij} is Hermitian. Equating the determinant of (24) to zero gives the dispersion relation for two normal waves,

$$N^4-(\eta_{xx}+\eta_{yy})N^2+\eta_{xx}\eta_{yy}-|\eta_{xy}|^2=0. \tag{25}$$

Solving (25) for N^2 gives the refractive indices N_1^2 and N_2^2 for slow (O-mode) and fast (X-mode) waves, respectively,

$$N_{1,2}^2=\frac{\eta_{xx}+\eta_{yy}}{2}\pm\frac{1}{2}\sqrt{R},\quad R=(\eta_{xx}-\eta_{yy})^2+4|\eta_{xy}|^2. \tag{26}$$

Function R determines the difference between N_1 and N_2 and, thus, the phase between two normal waves. In the cold plasma case [with dielectric tensor (16)], solutions (26) have the well known form

$$N_{1,2}^{(c)^2}=1-\frac{2X(1-X)}{2(1-X)-Y^2\sin^2\alpha\pm\sqrt{Q}},$$
$$Q=Y^4\sin^4\alpha+4Y^2(1-X)^2\cos^2\alpha. \tag{27}$$

The components of $\mathbf{E}$ are determined by the polarization factor $p=E_y/E_x$. Expressing N^2 in terms of p from the first equation (24), $N^2=\eta_{xx}+\eta_{xy}p$, and using (26) yields two solutions,

$$p_{1,2}=i(g\mp\sqrt{g^2+1}),\quad g=\frac{i(\eta_{xx}-\eta_{yy})}{2\eta_{xy}}. \tag{28}$$

The upper sign corresponds to the slow wave with $N_1>N_2$. Since g is pure real, the factor p is pure imaginary. Then, the two eigenvectors of the normal modes are as follows:

$$\mathbf{E}_1=\frac{1}{\sqrt{1+\Lambda^2}}\begin{pmatrix}i\\\Lambda\end{pmatrix},\quad \mathbf{E}_2=\frac{1}{\sqrt{1+\Lambda^2}}\begin{pmatrix}\Lambda\\i\end{pmatrix},$$
$$\Lambda=\sqrt{1+g^2}-g\quad(0\leqslant\Lambda\leqslant 1). \tag{29}$$

They describe electromagnetic waves elliptically polarized in mutually orthogonal directions. The major semiaxes of the ellipses are parallel to the x and y axes. Recall that the x axis was chosen along the perpendicular component of the magnetic field, $\mathbf{B}_{0\perp}=\mathbf{B}_0-\mathbf{e}_z(\mathbf{e}_z\cdot\mathbf{B}_0)$. This means that in spatially varying magnetic field, the major semiaxis of the slow wave follows the $\mathbf{B}_{0\perp}$ direction while the ellipse of the fast wave is elongated in the $\mathbf{B}_0\times\mathbf{e}_z$ direction. Introducing the laboratory reference frame, x_L,y_L,z_L, with the z_L axis parallel to the z axis and, correspondingly, to the wave propagation direction, the position of the ellipse is characterized by the angle ψ between the major axis and Ox_L $(0\leqslant\psi\leqslant\pi)$ (see Fig. 2). The ratio of minor and major semiaxis (ellipticity) is characterized by the variable χ,

$$\tan\chi=\pm\frac{b}{a}=\pm\Lambda,\quad b\leqslant a, \tag{30}$$

where the positive and negative signs are for anticlockwise and clockwise rotation, looking forward to the laser beam source $(-\pi/4\leqslant\chi\leqslant\pi/4)$.

The standard approach to the evolution of polarization is based on a presentation of the incident on plasma wave as a superposition of two normal waves at the plasma-vacuum interface. The wave dynamics at any other position is determined by the evolution of the normal waves whose eigenvectors and the phases follow the above spatial dependences.

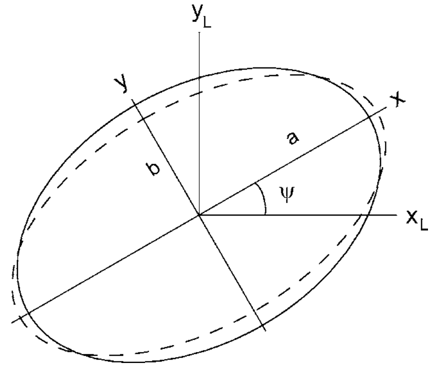

FIG. 3. Polarization ellipse for the electric field vector of the slow wave (O-mode). Solid line corresponds to $\tau=0$, dashed line illustrates the effect of thermal corrections at $\tau=0.06$ ($Y=0.01$, $\cos\alpha=0.01$). The polarization ellipse of the fast wave (X-mode) is similar but oriented in the **y** direction.

Different applications of this approach for cold plasma are discussed in the literature (see, for example, Ref. 13). We focus our attention on the finite electron temperature effects considering this problem perturbatively. Instead of direct power expansion of (26) and (29) in τ, we apply an alternative approach based on perturbative treatment of the Jones matrix (24). For this purpose, the tensor η_{ij} is presented as a sum of the cold part (B1) and a small term (B2) proportional to τ. The results of the perturbative analysis for the refractive indices and the polarization vectors are as follows:

$$N_{1,2}^2 = N_{1,2}^{(c)^2} + \tau X\left(\frac{5}{2} - N^2 \pm \frac{2(3N^2-5)Y\cos^2\alpha + Y^3\sin^4\alpha(6N^2-15/4)}{\sqrt{Y^2\sin^4\alpha + 4\cos^2\alpha}}\right), \quad (31)$$

$$\mathbf{E}_1 = \frac{1}{\sqrt{1+\Lambda^{(c)^2}}}\begin{pmatrix} i(1+\tau\Lambda^{(c)^2}D) \\ \Lambda^{(c)}(1-\tau D)\end{pmatrix},$$
$$\mathbf{E}_2 = \frac{1}{\sqrt{1+\Lambda^{(c)^2}}}\begin{pmatrix} \Lambda^{(c)}(1-\tau D) \\ i(1+\tau\Lambda^{(c)}D)\end{pmatrix}, \quad (32)$$

where

$$D = \left(9N^2 - \frac{5}{2}\right)\frac{1-\Lambda^{(c)^2}}{(1+\Lambda^{(c)^2})^2}, \quad \Lambda^{(c)} = \sqrt{g^{(c)^2}+1} - g^{(c)},$$
$$g^{(c)} = \frac{Y\sin^2\alpha}{2(1-X)\cos\alpha}. \quad (33)$$

From (32) and (33), it follows that for pure parallel and perpendicular propagation ($\Lambda^{(c)}=1$ and $\Lambda^{(c)}=0$, respectively), the thermal corrections do not change the polarization properties in comparison with the cold plasma case. Specifically, if $\sin\alpha=0$, the waves are circularly polarized while at $\cos\alpha=0$ the polarization is linear. In the intermediate case of elliptical polarization, the effect of thermal corrections is illustrated in Fig. 3. It results in a squeezing of the ellipses (decrease of ellipticity $l=b/a$) in comparison with the cold plasma case. Corresponding change $\Delta l^{(T)}$ of the ellipticity is given by the equation

$$\Delta l^{(T)} = -\tau D(1+\Lambda^{(c)^2})l^{(c)}, \quad l^{(c)} = \Lambda^{(c)}. \quad (34)$$

Equation (31) describes the influence of electron thermal motion on the dispersion of two characteristic waves in the weakly relativistic limit. Focusing on interferometric applications, we ignore the birefringence of two normal waves caused by the magnetic field. Introducing the interferometric phase Φ as the phase difference between laser beams passing through vacuum and plasma,

$$\Phi = \int_v k_v dz - \int_p k_p dz, \quad (35)$$

and decomposing $\Phi=\Phi^{(c)}+\Delta\Phi^{(T)}$, yields the relative change of Φ caused by the thermal effects,

$$\Delta\Phi^{(T)}/\Phi^{(c)} = \left(N^2 - \frac{5}{2}\right)\int_p \tau X dz \bigg/ \int_p X dz$$
$$\simeq \left(N^2 - \frac{5}{2}\right)\tau \simeq -\frac{3}{2}\tau. \quad (36)$$

The positive factor $N^2\simeq 1$ results from the nonrelativistic Doppler mechanism and leads to the increase of Φ. The weakly relativistic factor $-5/2$ changes the sign of $\Delta\Phi^{(T)}$ resulting in a decrease of the interferometric phase in comparison with the cold plasma case. The effect is caused by the relativistic mass of the electrons.

Finite magnetic field introduces anisotropy and birefringence into the normal wave propagation. The difference of the refractive indices is determined mainly by the first term in Eq. (31) caused by the cold plasma response. At finite electron temperature, there is also a contribution from the second temperature-dependent term. This makes the evolution of polarization sensitive to T_e. Specifically, for propagation at the angle α not too close to 90° ($\cos\alpha \gg Y$), the difference between N_1 and N_2 is linear in Y and determined by $|\eta_{xy}|$,

$$N_1 - N_2 = XY\cos\alpha[1+\tau(3N^2-5)]. \quad (37)$$

Correspondingly, the Faraday rotation angle

$$\Theta = \frac{\omega}{2c}\int (N_1 - N_2)dz \quad (38)$$

is a sum of rotation angle in cold plasma $\Theta^{(c)}$ and small thermal correction $\Delta\Theta^{(T)}$. The magnitude of the relative change of Θ is represented by the ratio

$$\Delta\Theta^{(T)}/\Theta^{(c)} = (3N^2-5)\int_p \tau XY\cos\alpha dz \bigg/ \int_p XY\cos\alpha dz$$
$$\simeq (3N^2-5)\tau \simeq -2\tau. \quad (39)$$

The positive nonrelativistic factor $3N^2\simeq 3$ originates from the Doppler effect. The factor -5 is caused by the weakly relativistic effects. It changes the sign of the thermal correction and, thus, decreases the value of Θ relative to the cold plasma case.

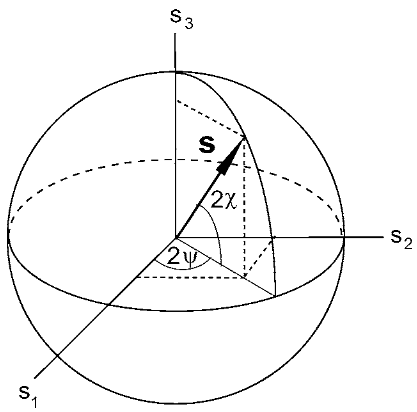

FIG. 4. The Cartesian coordinates of the polarization Stokes vector **s** in terms of angular positions (ψ,χ) on the Poincare sphere.

For the quasiperpendicular case, $\cos\alpha \ll Y$, the difference between two refractive indices is smaller, $\propto Y^2$, and the polarization evolves accordingly to the Cotton-Mouton effect. In this case, the factor $\eta_{xx}-\eta_{yy}\propto XY^2$ determines the evolution of polarization. From Eq. (31), it follows that the relative change of the phase between two normal waves is equal to $\tau(12N^2-15/2)\simeq 9\tau/2$. The weakly relativistic factor $-15/2$ reduces the nonrelativistic dispersive term, $12N^2 \simeq 12$, but not enough to change the sign. For the general case of arbitrary propagation, the difference between N_1 and N_2 is determined by a contribution to R from both $|\eta_{xy}|$ and $\eta_{xx}-\eta_{yy}$ factors. In this situation, the Faraday and the Cotton-Mouton effects are mixed together.

There is a presentation for the evolution of polarization where the contributions from these two effects are formally separated. It is based on the differential equation formulated in terms of the evolution of the Stokes vector of polarization **s**.[14] The three-component Stokes vector $\mathbf{s}=(s_1,s_2,s_3)$ is a unit vector, $s_1^2+s_2^2+s_3^2=1$, whose Cartesian components are related to (ψ,χ) as follows:

$$s_1=\cos 2\chi\cos 2\psi,\quad s_2=\cos 2\chi\sin 2\psi,\quad s_3=\sin 2\chi, \tag{40}$$

where ψ is the azimuth of the polarization ellipse; χ is defined by (30) (see Fig. 4). According to Ref. 14, the equation of evolution has the form

$$\frac{d\mathbf{s}}{dz}=\mathbf{\Omega}\times\mathbf{s} \tag{41}$$

and describes rotation of the Stokes vector **s** around the vector of the angular velocity $\mathbf{\Omega}$,

$$\mathbf{\Omega}=\frac{\omega}{c}(N_1-N_2)\mathbf{s}_1, \tag{42}$$

where $\mathbf{s}_1$ is a Stokes vector of the slow wave with $N_1>N_2$. Using Eqs. (29) and (30) yields trigonometric functions for s_1 and s_2,

$$\cos 2\chi=\frac{g}{\sqrt{g^2+1}},\quad \sin 2\chi=\frac{1}{\sqrt{g^2+1}}. \tag{43}$$

The difference of the refractive indices is approximated from Eq. (26) at $X\ll 1$,

$$N_1-N_2=\frac{\sqrt{R}}{N_1+N_2}\simeq -i\eta_{xy}\sqrt{g^2+1}. \tag{44}$$

Combining Eq. (43) with Eq. (44) gives the components of $\mathbf{\Omega}$,

$$\mathbf{\Omega}=\frac{\omega}{2c}\begin{pmatrix}(\eta_{xx}-\eta_{yy})\cos 2\psi\\(\eta_{xx}-\eta_{yy})\sin 2\psi\\-2i\eta_{xy}\end{pmatrix}. \tag{45}$$

The first two components are related to the Cotton-Mouton factor $\eta_{xx}-\eta_{yy}$ while the third one is proportional to the Faraday factor η_{xy}. Using Eq. (B1) for $\eta_{ij}^{(c)}$ and Eq. (B2) for $\Delta\eta_{ij}^{(T)}$ allows us to present vector $\mathbf{\Omega}$ in the form $\mathbf{\Omega}=\mathbf{\Omega}^{(c)}+\Delta\mathbf{\Omega}^{(T)}$, where $\mathbf{\Omega}^{(c)}$ is the contribution from the cold plasma and $\Delta\mathbf{\Omega}^{(T)}$ is linear in T_e thermal correction,

$$\mathbf{\Omega}^{(c)}=\frac{\omega X}{2Zc}\begin{pmatrix}Y^2\sin^2\alpha\cos 2\psi\\Y^2\sin^2\alpha\sin 2\psi\\2Y(1-X)\cos\alpha\end{pmatrix},$$
$$\Delta\mathbf{\Omega}^{(T)}=\frac{3\omega X\tau}{c}\begin{pmatrix}(2N^2-5/4)Y^2\sin^2\alpha\cos 2\psi\\(2N^2-5/4)Y^2\sin^2\alpha\sin 2\psi\\(N^2-5/3)Y\cos\alpha\end{pmatrix}. \tag{46}$$

Proportional to N^2 terms in $\Delta\mathbf{\Omega}^{(T)}$ coincide at $N^2=1$ with the factors calculated in Ref. 4. Additional factors $-5/4$, $-5/4$, and $-5/3$ originate from the relativistic electron mass increase. The dominant component $\Delta\Omega_3^{(T)}$ is responsible for the Faraday effect. The relativistic factor $-5/3$ makes this component negative, indicating that the effect of finite electron temperature reduces the Faraday rotation angle in comparison with the cold plasma case.

IV. DISCUSSION

The results presented in this paper show that the combination of both nonrelativistic and relativistic mechanisms leads to a relative change of the interferometric phase, Faraday rotation angle, and Cotton-Mouton effect by factors $\tau(N^2-5/2)$, $\tau(3N^2-5)$, and $\tau(12N^2-15/2)$, respectively. The factors $N^2\simeq 1$, $3N^2\simeq 3$, and $12N^2\simeq 12$ ($N\simeq 1$ for a high-frequency electromagnetic wave) were found in Ref. 4 while the additional factors $-5/2$, -5, and $-5/2$ result from the relativistic effects derived herein. Thus, both nonrelativistic thermal corrections and weakly relativistic thermal effects are essential to correctly interpret interferometric measurements in a high-temperature plasma. For plasma with $T_e \simeq 10$ keV, the thermal corrections are -3% for the interferometric phase, -4% for the Faraday effect, and 9% for the Cotton-Mouton effect. Note that previous calculations performed without relativistic effects predict a 2%, 6%, and 24% increase for the interferometric phase, the Faraday rotation angle, and the Cotton-Mouton effect, respectively.[4]

To express the effect of finite electron temperature in practical units, let us consider, for example, the Faraday rotation measurements. In cold plasma, the angle of rotation Θ_c is determined by the equation

$$\Theta_c(\text{rad}) = 2.62 \times 10^{-25}\lambda^2(\mu m^2) \int n(m^{-3})B_z(T)dz(m). \quad (47)$$

For typical far-infrared (FIR) laser wavelength $\lambda = 100\ \mu$m and plasma parameters, $B=1$ T, $n=10^{20}$ m^{-3}, $T_e \simeq 10$ keV, $z \simeq 1$ m, an estimate of the thermal correction $\Delta\Theta^{(T)}$ to the Faraday rotation angle in cold plasma, $\Theta^{(c)} \simeq 15°$, is as follows:

$$\Delta\Theta^{(T)} = -2(T_e/m_e c^2)\Theta^{(c)} \simeq 0.6^0. \quad (48)$$

This is much larger than the typical system noise levels in the range $\simeq 0.01°$ (Ref. 15) to $0.04°$.[16] Experimental observation of a negative thermal correction caused by the relativistic effects would be an important verification of fundamental relativistic physics in high-temperature plasma devices. In addition, the thermal correction to interferometric phase measurements can potentially be exploited to measure the electron temperature in future reactor relevant devices.

Current design of density measurements by both interferometery and polarimetry is based on the wave dispersion relation in a cold plasma (i.e., $T_e=0$). ITER will operate at electron temperatures in the range $T_e \sim 10$–25 keV. Linear temperature corrections decrease the refractive indices due to the nonrelativistic Doppler shift mechanism and increase them due to the relativistic mass effect. Since the relativistic factors are larger and of opposite sign to nonrelativistic factors for the interferometric phase and the Faraday rotation angle, the measured phase is reduced as compared to a cold plasma as given by

$$\Phi = C_I \int n_e\left(1 - \frac{3}{2}\frac{T_e}{m_e c^2}\right)dz, \quad (49)$$

$$\Theta = C_F \int n_e B_z\left(1 - 2\frac{T_e}{m_e c^2}\right)dz, \quad (50)$$

where C_I and C_F are constants. This means that without thermal corrections, the interferometer will underestimate the density while the polarimeter will lead to an underestimate of the magnetic field. However, since electron temperature is known from Thomson scattering, finite T_e effects can be corrected. Alternatively, for a tangentially viewing interferometer-polarimeter system, the toroidal field is largely known, as in the case for a tokamak. Then the above two equations have only two unknowns and hence provide information on both the plasma electron density and temperature.

In order to provide a numerical check on this analytic result, calculations using the GENRAY[17] ray tracing code have been carried out. GENRAY is a general ray tracing code for the calculation of electromagnetic wave propagation and absorption in the geometrical optics approximation that can be applied to tokamak equilibria. The code offers several alternative dispersion functions, and for this study the fully relativistic electron plasma option described in Ref. 18 was employed. The magnetic field geometry is specified to be that of a predicted ITER scenario 2 equilibrium with $I_p = 15$ MA, $R=6.2$ m, and $B_T=5.3$ T. Both the electron density and temperature profiles are assumed to be flat with $n_e = 10^{20}$ m^{-3} and T_e in the range 1–25 keV. The actual frequency and ray trajectory used corresponds to a tangentially viewing $\lambda=100\ \mu$m probe beam along the ITER midplane with a tangency radius of $R=6.8$ m.

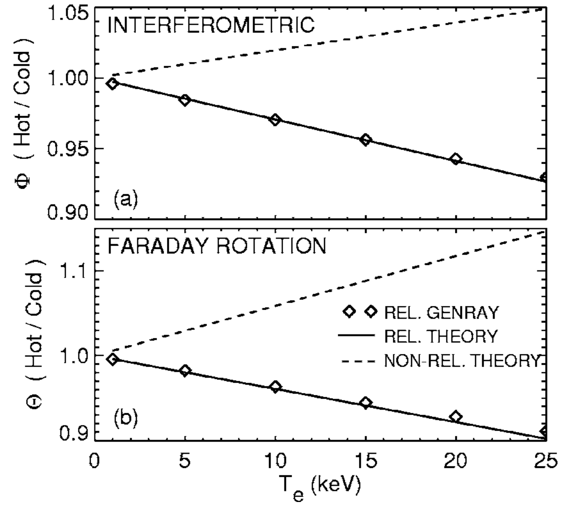

FIG. 5. Plots of weakly relativistic analytical results for the interferometric phase $1+\Delta\Phi^{(T)}/\Phi^{(c)}$ given by (36) [(a) solid line] and for the Faraday angle $1+\Delta\Theta^{(T)}/\Theta^{(c)}$ given by (39) [(b) solid line] in comparison with corresponding fully relativistic (diamond) and nonrelativistic (dashed line) ray tracing code (GENRAY) computations.

For comparison to analytical results (36) and (39), both X and O mode rays were launched, and the phase shift (35) of each ray relative to vacuum propagation was calculated using the standard cold-plasma dispersion relation as well as the fully relativistic one. The interferometric phase shift and the Faraday rotation angle are then defined from the GENRAY calculations as

$$\Phi = \tfrac{1}{2}(\Phi_X + \Phi_O), \quad \Theta = \tfrac{1}{2}(\Phi_X - \Phi_O). \quad (51)$$

Shown in Figs. 5(a) and 5(b) are the GENRAY calculated interferometeric phase shifts [(a), diamond] and Faraday rotation angles [(b), diamond] for a range of electron temperatures, where the hot plasma results have been scaled to that of the cold plasma. Also shown in Fig. 5 is the analytic prediction from Eqs. (36) and (39) (solid), as well as the nonrelativistic thermal plasma correction from Ref. 18 (dashed), consistent with Ref. 4. It is obvious that both weakly relativistic nondispersive and nonrelativistic dispersive thermal corrections play a role in determining the overall ray dispersion and that these effects are approximated well by Eqs. (36) and (39). It is important to point out that calculations have been carried out for a variety of tangency radii and wavelengths with the same level of agreement in all cases.

ACKNOWLEDGMENTS

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APPENDIX A: TEMPERATURE CORRECTIONS TO THE DIELECTRIC TENSOR

Equation (4) for δf is solved iteratively by using small parameter $\omega_{ce}/\omega \ll 1$. First, we introduce a new function,

$$\delta g = \delta f \exp \Psi. \tag{A1}$$

Variable Ψ is defined by the integral in the spherical reference frame (p,θ,ϕ) (see Sec. II),

$$\Psi = i\int_0^{\phi} \frac{\mathbf{k}\cdot\mathbf{p}}{m\omega_{ce}} d\phi' = iq(\phi \cos\alpha \cos\theta + \sin\alpha \sin\theta \sin\phi),$$
$$q = \frac{kp}{m\omega_{ce}} \simeq k\rho_{Le}. \tag{A2}$$

Then, δg can be rewritten as

$$\delta g = \epsilon \frac{\partial \delta g}{\partial \phi} + R, \tag{A3}$$

where

$$R = -\frac{ie}{p\omega}(\mathbf{E}\cdot\mathbf{p})\frac{\partial f}{\partial p}\exp\Psi, \quad \epsilon = -\frac{iY}{\gamma}, \quad Y = \frac{\omega_{ce}}{\omega}. \tag{A4}$$

As a first step of iteration, we set to zero the small term $\propto \epsilon$ and obtain the zero-order solution

$$\delta g_0 = R = -\frac{ie}{\omega}\mathcal{E}\frac{\partial f}{\partial p}\exp\Psi,$$
$$\mathcal{E}(\phi) = E_x \sin\theta\cos\phi + E_y \sin\theta \sin\phi + E_z \cos\theta. \tag{A5}$$

Next-order corrections are obtained by making power series expansion in ϵ,

$$\delta g = \sum_{n=0}^{n=\infty} \delta g_n, \quad \delta g_n \propto \epsilon^n, \quad n = 0,1,2,\ldots\ . \tag{A6}$$

Substituting (A6) into (A3) and combining terms of the same order in ϵ yields the recursion equation that allows us to calculate the next-order correction by differentiating the previous one,

$$\delta g_{i+1} = \epsilon \frac{\partial \delta g_i}{\partial \phi}, \quad i = 1,2,\ldots\ . \tag{A7}$$

Small parameter $\epsilon \simeq 10^{-2}$ provides good convergence of the series. Fast oscillating harmonics $\sin n\phi$ and $\cos n\phi$ can slow down the convergence at large $n \simeq \epsilon^{-1}$. Contributions from these terms are small, and, therefore, ignored below.

The solution for δg is presented by the series

$$\delta g = R + \epsilon \frac{\partial R}{\partial \phi} + \cdots + \epsilon^n \frac{\partial^n R}{\partial \phi^n} + \cdots. \tag{A8}$$

A similar expansion for δf has the form

$$\delta f = -\frac{ie}{\omega}\left(\sum_{n=0} \epsilon^n Q_n\right)\frac{\partial f}{\partial p}, \quad Q_n = \exp(-\Psi)\frac{\partial^n}{\partial \phi^n}(\mathcal{E}\exp\Psi). \tag{A9}$$

The angular dependences of δf are described by the Q_n factors. To illustrate the structure of Q_n, we present the first five terms of (A9) (up to ϵ^4 order),

$$\begin{aligned}\delta f(p,\theta,\phi) \propto{}& \mathcal{E} + \epsilon[\mathcal{E}' + \mathcal{E}\Psi'] + \epsilon^2[\mathcal{E}'' + 2\mathcal{E}'\Psi' + \mathcal{E}(\underline{\Psi'^2} \\ &+ \Psi'')] + \epsilon^3[\mathcal{E}''' + 3\mathcal{E}''\Psi' + 3\mathcal{E}'(\Psi'' + \underline{\Psi'^2}) \\ &+ \mathcal{E}(\Psi'^3 + 3\underline{\Psi'\Psi''} + \Psi''')] + \epsilon^4[\mathcal{E}'''' + 4\mathcal{E}'''\Psi' \\ &+ 6\mathcal{E}''(\Psi'' + \underline{\Psi'^2}) + 4\mathcal{E}'(\Psi'^3 + 3\underline{\Psi'\Psi''} + \Psi''') \\ &+ \mathcal{E}(\underline{\Psi'^4} + 6\Psi'^2\Psi'' + 3\underline{\Psi''^2} + 4\underline{\Psi'\Psi'''} + \Psi'''')].\end{aligned} \tag{A10}$$

The terms containing derivatives of Ψ are proportional to the corresponding powers of $q \propto k$ and, thus, represent the dispersive thermal corrections. In addition to this, each factor Q_n has one term, $\partial^n \mathcal{E}/\partial^n \phi$, that does not depend on k and represents the nondispersive contribution. Selecting only these terms yields an infinite series for the nondispersive part of δf,

$$\delta f^{(ND)} = -\frac{ie}{\omega}\frac{\partial f}{\partial p}\sum_{n=0} \epsilon^n \frac{\partial^n \mathcal{E}}{\partial \phi^n}. \tag{A11}$$

After integration over θ and ϕ according to Eq. (5), the sum (A11) is calculated exactly. We refer to the result of this summation as the nondispersive (ND) part of the plasma conductivity tensor.

(i) Nondispersive part of the plasma response. Elements of the nondispersive conductivity tensor are presented by the integrals over p,

$$\begin{pmatrix} j_{x'}^{(ND)} \\ j_{y'}^{(ND)} \\ j_{z'}^{(ND)} \end{pmatrix} = -\frac{i\omega_{pe}^2}{3\omega}\int_0^{\infty} \frac{p^3 dp}{\gamma(1 - Y^2/\gamma^2)}\frac{\partial f}{\partial p}\begin{pmatrix} 1 & -iY/\gamma & 0 \\ iY/\gamma & 1 & 0 \\ 0 & 0 & 1 - Y^2/\gamma^2 \end{pmatrix}\begin{pmatrix} E_{x'} \\ E_{y'} \\ E_{z'} \end{pmatrix}. \tag{A12}$$

Matrix (A12) has a similarity to the dielectric tensor in cold plasma (11). In contrast to Eq. (11), expression (A12) contains integration over p and momentum-dependent factors $\gamma(p)$. This yields the dependence of the nondispersive part on the electron temperature. Considering a weakly relativistic limit, we divide Eq. (A12) into two parts: (i) the cold plasma tensor and (ii) a first-order correction proportional to τ. To accomplish this, Eq. (A12) is integrated by parts with the use of $df=(\partial f/\partial p)dp$. Differentiating over p in the resulting integrands yields three different derivatives. Using the weakly relativistic approximation $\gamma\simeq 1+p^2/2m^2c^2$, $d\gamma/dp \simeq p/m^2c^2$ allows us to simplify these derivatives,

$$\frac{d}{dp}\left(\frac{\gamma p^3}{\gamma^2-Y^2}\right)\simeq\frac{3p^2}{1-Y^2}\left(1-\frac{1+Y^2}{1-Y^2}\frac{5p^2}{6m^2c^2}\right),$$

$$\frac{d}{dp}\left(\frac{p^3}{\gamma^2-Y^2}\right)\simeq\frac{3p^2}{1-Y^2}\left(1-\frac{1}{1-Y^2}\frac{5p^2}{3m^2c^2}\right),\tag{A13}$$

$$\frac{d}{dp}\left(\frac{p^3}{\gamma}\right)\simeq 3p^2\left(1-\frac{5p^2}{6m^2c^2}\right).$$

Then, the results of integration by parts are presented by three integrals over p. They have similar forms,

$$\int_0^\infty\left(1-\frac{ap^2}{m^2c^2}\right)f(p)p^2dp,$$
$$a=(5/6)(1+Y^2)/(1-Y^2),\quad (5/3)(1-Y^2)^{-1},\quad 5/6,\tag{A14}$$

with three different constants a. The contribution from the first proportional to unity term in (A14) yields the factor $1/4\pi$ due to the normalization condition (6). The corresponding part of (A12) represents the dielectric tensor (11) of cold magnetized plasma. Integrating the second $\propto p^2$ term, one can use a nonrelativistic Maxwellian distribution function $f\propto\exp(-p^2/2m_eT_e)$. This yields integrals that determine the mean thermal energy of nonrelativistic Maxwellian gas, $\langle p^2/2m_e\rangle=3T_e/2$. Finally, the nondispersive part of ϵ'_{ij} is presented by Eq. (12).

(ii) Dispersive part of the dielectric tensor. Consider the contribution to Q_n from q-dependent dispersive terms. At given n they are presented by a polynomial of degree n, $p_n=\Sigma_{k=1}^{k=n}c_kq^k$, where c_k are the products of the trigonometric functions of ϕ and θ. Integrating p_n over θ and ϕ accordingly to (5) shows that only even powers of q contribute to $\mathbf{j}$ while terms with odd powers cancel after integration. All nonvanishing dispersive terms are underlined in Eq. (A10).

Using definition (A2), the characteristic value of q takes the form

$$q\simeq\frac{\sqrt{\tau}N}{Y}\tag{A15}$$

leading to the estimation $5\lesssim q\lesssim 20$. The contribution from the dispersive terms to (A10) can be schematically written as follows:

$$\delta f^{(D)}\propto\tau N^2+i\tau YN^2+\tau(Y^2N^2+\tau N^4)+\cdots.\tag{A16}$$

The convergence of the series (A16) is provided at small $\tau\ll 1$ and $Y\ll 1$. The first term, proportional to τN^2, determines the thermal correction to the refractive indices. The imaginary term contributes to the off-diagonal elements of ϵ_{ij} responsible for the Faraday effect. The small third factor originates from ϵ^4-order terms in Eq. (A10). This factor is important for correct treatment of the Cotton-Mouton effect. It consists of two parts. The first, proportional to the τY^2N^2 factor, determines thermal correction to the Cotton-Mouton effect. The second term, quadratic in τ, is formally larger than the first one but eventually cancels out. This term yields equal contributions to the diagonal η_{xx} and η_{yy} elements of the Jones matrix. Since these factors are canceled in the combination $\eta_{xx}-\eta_{yy}$ that determines the dynamics of the Cotton-Mouton effect, they play no role in the evolution of polarization. Taking into account the aforementioned terms and ignoring higher-order corrections yields the dispersive part of the distribution function $\delta f^{(D)}$,

$$\delta f^{(D)}(p,\theta,\phi)=-\frac{ie}{\omega}\frac{\partial f}{\partial p}[\epsilon^2\mathcal{E}\Psi'^2+3\epsilon^3(\mathcal{E}'\Psi'^2+\mathcal{E}\Psi'\Psi'')+\epsilon^4(6\mathcal{E}''\Psi'^2+12\mathcal{E}'\Psi'\Psi''+3\mathcal{E}\Psi''^2+4\mathcal{E}\Psi'\Psi''')].\tag{A17}$$

Expression (A17) determines linear in τ dispersive corrections to ϵ'_{ij} to the lowest ϵ^4 order needed for the polarization analysis. Calculations of the angular dependences of $\delta f^{(D)}$ and integrals (5) are straightforward. After integrations over θ and ϕ, the dispersive part of the conductivity tensor is determined by the integrals over p,

$$\begin{pmatrix}j_{x'}^{(D)}\\ j_{y'}^{(D)}\\ j_{z'}^{(D)}\end{pmatrix}=-\frac{i\omega_{pe}^2}{15\omega}\frac{k^2c^2}{\omega^2}\int_0^\infty\frac{p^5dp}{m^2c^2}\frac{\partial f}{\partial p}\begin{pmatrix}1+6Y^2+(2+9Y^2)\sin^2\alpha & -3iY(1+\sin^2\alpha) & (1+Y^2)\sin 2\alpha\\ 3iY(1+\sin^2\alpha) & 1+Y^2(6+7\sin^2\alpha) & 3iY\sin 2\alpha/2\\ (1+Y^2)\sin 2\alpha & -3iY\sin 2\alpha/2 & 1+2\cos^2\alpha+Y^2\sin^2\alpha\end{pmatrix}\begin{pmatrix}E_{x'}\\ E_{y'}\\ E_{z'}\end{pmatrix}.\tag{A18}$$

Since the integrand is proportional to p^5, the lowest-order term in power expansion of Eq. (A18) in $\tau\ll 1$ is proportional to T_e. Correspondingly, the nonrelativistic version of (A17) with $\gamma=1$ and the nonrelativistic Maxwellian distribution function $f\propto\exp(-p^2/2mT_e)$ is used in (A18). Integrating by parts yields integrals proportional to the mean energy of nonrelativistic Maxwellian gas, $\langle p^2/2m_e\rangle=3T_e/2$. Finally, the dispersive part of ϵ'_{ij} is given by Eq. (13).

APPENDIX B: THERMAL EFFECTS ON THE NORMAL MODES

The Jones equations (24) determine refractive indices and normal vectors $\mathbf{E}_1$ and $\mathbf{E}_2$ for slow (O-mode) and fast (X-mode) electromagnetic waves. The corresponding solutions in cold plasma are well known. We treat the effects of electron thermal motion perturbatively. The Jones tensor η_{ij} is presented as a sum of cold part $\eta_{ij}^{(c)}$ and linear in τ correction $\Delta\eta_{ij}^{(T)}$. The cold part is calculated with the use of $\epsilon_{ij}^{(c)}$ given by Eq. (16),

$$\eta_{xx}^{(c)} = 1 - \frac{X(1-X)}{Z} + \frac{XY^2\sin^2\alpha}{Z},$$

$$\eta_{yy}^{(c)} = 1 - \frac{X(1-X)}{Z}, \quad \eta_{xy}^{(c)} = \frac{i(1-X)XY\cos\alpha}{Z}, \tag{B1}$$

$$\eta_{xx}^{(c)} - \eta_{yy}^{(c)} = \frac{XY^2\sin^2\alpha}{Z},$$

$$Z = (1-X)(1-Y^2) - XY^2\sin^2\alpha.$$

The elements of $\Delta\eta_{ij}^{(T)}$ are determined by the thermal corrections (18) and (21). Since the dispersive part $\epsilon_{ij}^{(D)}$ is found with Y^2 accuracy, we keep the same accuracy in calculation of $\Delta\eta_{ij}^{(T)}$. This allows us to approximate $\Delta\eta_{xx}^{(T)} \simeq \Delta\epsilon_{xx}^{(T)}$ and $\Delta\eta_{xy}^{(T)} \simeq \Delta\epsilon_{xy}^{(T)}$. But for $\Delta\eta_{yy}^{(T)}$, this reduction requires both $Y \ll 1$ and $X \ll 1$. The analysis of arbitrary X is straightforward but leads to longer expressions so that we present calculations of $\Delta\eta_{ij}^{(T)}$ to the leading order in $X \ll 1$, yielding

$$\Delta\eta_{ij}^{(T)} = \tau X\begin{pmatrix} 5/2 - N^2 + 3Y^2(5/2-2N^2) & iY\cos\alpha(3N^2-5) \\ -iY\cos\alpha(3N^2-5) & 5/2 - N^2 + 3Y^2(5/2-2N^2)\end{pmatrix} - \tau XY^2\sin^2\alpha\begin{pmatrix} 15/2 - 5N^2 & 0 \\ 0 & 7N^2\end{pmatrix}. \tag{B2}$$

The Jones equations (24) are rewritten in a compact form,

$$(\eta + \delta\eta)\cdot\mathbf{E} = \mu\mathbf{E}, \quad \mu = N^2, \quad \mathbf{E} = \begin{pmatrix} E_x \\ E_y \end{pmatrix}, \tag{B3}$$

where η and $\delta\eta$ are represented by (B1) and (B2), respectively. We expand solutions in powers of $\delta\eta \propto \tau$,

$$\mathbf{E} = \mathbf{E}^{(c)} + \mathbf{E}^{(T)} + \cdots, \quad \mu = \mu^{(c)} + \mu^{(T)} + \cdots, \tag{B4}$$

where zero-order quantities $\mu^{(c)}$ are given by (27) (our notation $\mu = N^2$ is different from the standard one, $\mu = N$), while $\mathbf{E}^{(c)}$ follows from (29) at $\tau = 0$,

$$\mathbf{E}_1^{(c)} = \frac{1}{\sqrt{1+\Lambda^{(c)2}}}\begin{pmatrix} i \\ \Lambda^{(c)}\end{pmatrix}, \quad \mathbf{E}_2^{(c)} = \frac{1}{\sqrt{1+\Lambda^{(c)2}}}\begin{pmatrix} \Lambda^{(c)} \\ i\end{pmatrix}, \tag{B5}$$

where

$$\Lambda^{(c)} = \sqrt{1+g^{(c)2}} - g^{(c)}, \quad g^{(c)} = \frac{Y\sin^2\alpha}{2(1-X)\cos\alpha}. \tag{B6}$$

Vectors $\mathbf{E}_\alpha^{(c)}$ form the orthogonal basis

$$\mathbf{E}_\alpha^{(c)\star}\mathbf{E}_\beta^{(c)} = \begin{cases} 1, & \alpha = \beta \\ 0, & \alpha \neq \beta, \end{cases} \tag{B7}$$

where

$$\mathbf{E}_1^{(c)\star} = \frac{1}{\sqrt{1+\Lambda^{(c)2}}}(-i, \Lambda^{(c)}), \quad \mathbf{E}_2^{(c)\star} = \frac{1}{\sqrt{1+\Lambda^{(c)2}}}(\Lambda^{(c)}, -i). \tag{B8}$$

The perturbed electric field is presented by a superposition of cold plasma normal modes [Eq. (B5)],

$$\mathbf{E} = \sum_{\alpha=1}^{\alpha=2} c_\alpha \mathbf{E}_\alpha^{(c)}. \tag{B9}$$

Substituting Eq. (B9) in Eq. (B3) and taking into account that $\eta\cdot\mathbf{E}_\alpha^{(c)} = \mu_\alpha^{(c)}\mathbf{E}_\alpha^{(c)}$ yields equations for c_α,

$$\sum_{\alpha=1}^{\alpha=2} c_\alpha(\mu - \mu_\alpha^{(c)})\mathbf{E}_\alpha^{(c)} = \sum_{\alpha=1}^{\alpha=2} c_\alpha\delta\eta\cdot\mathbf{E}_\alpha^{(c)}. \tag{B10}$$

Upon multiplying Eq. (B10) by $\mathbf{E}_\beta^\star$, these equations become

$$(\mu - \mu_\beta^{(c)})c_\beta = \sum_{\alpha=1}^{\alpha=2} c_\alpha \mathbf{E}_\beta^{(c)\star}\cdot\delta\eta\cdot\mathbf{E}_\alpha^{(c)}, \quad \beta = 1,2. \tag{B11}$$

The coefficients c_α are calculated perturbatively by power expansion in $\delta\eta \propto \tau$ of the form $c_\alpha = c_\alpha^{(c)} + c_\alpha^{(T)} + \cdots$. The pair of zero-order solutions $c_\alpha^{(c)}$ is determined by the unperturbed state of polarization (without thermal corrections). Considering, for example, the slow wave marked by "1," one should put $c_{s1}^{(c)} = 1$, $c_{s2}^{(c)} = 0$ (an additional index "s" is added to specify the choice of zero-order iteration). Since the right-hand side of Eq. (B11) is proportional to $\delta\eta$, the values of $c^{(\alpha)}$ in this term are determined by zero-order combination (1, 0). Then, the first equation (B11) with $\beta = 1$ yields the thermal correction to $\mu_1^{(c)}$,

$$\mu_1^{(T)} = \mathbf{E}_1^{(c)\star}\cdot\delta\eta\cdot\mathbf{E}_1^{(c)}, \tag{B12}$$

while the second equation with $\beta = 2$ gives the first-order correction $c_{s2}^{(T)}$,

$$c_{s2}^{(T)} = \frac{\mathbf{E}_2^{(c)\star}\cdot\delta\eta\cdot\mathbf{E}_1^{(c)}}{\mu_1^{(c)} - \mu_2^{(c)}}. \tag{B13}$$

The coefficient $c_{s1}^{(T)}$ in superposition (B9) is not determined by Eq. (B11). It is found from the normalization condition $\mathbf{E}_1^\star \cdot \mathbf{E}_1 = 1$ for the perturbed eigenvector (B4) and turns out to be of the second order in τ, so that we put $c_{s1}^{(T)}=0$. Similar corrections are valid for the fast wave,

$$\mu_2^{(T)} = \mathbf{E}_2^{(c)\star} \cdot \delta\eta \cdot \mathbf{E}_2^{(c)}, \quad c_{f1}^{(T)} = \frac{\mathbf{E}_1^{(c)\star} \cdot \delta\eta \cdot \mathbf{E}_2^{(c)}}{\mu_2^{(c)} - \mu_1^{(c)}}, \quad c_{f2} = 0. \tag{B14}$$

Calculations of the explicit expressions for $\mu_{1,2}^{(T)}$ and $c_{s1,f2}^{(T)}$ are straightforward,

$$\mu_{1,2}^{(T)} = \tau X \left[\frac{5}{2} - N^2 + \frac{Y^2}{8}[45 - 52N^2 + (15 + 4N^2)\cos 2\alpha] \pm 2(3N^2 - 5)\frac{Y \cos \alpha \Lambda^{(c)}}{1 + \Lambda^{(c)^2}} \pm \frac{3}{4} Y^2 \sin^2 \alpha (8N^2 - 5) \frac{1 - \Lambda^{(c)^2}}{1 + \Lambda^{(c)^2}} \right],$$

$$c_{s2}^{(1)} = c_{f1}^{(1)} = \frac{i\tau XY}{\mu_1^{(c)} - \mu_2^{(c)}} \left[(5 - 3N^2)\cos \alpha \frac{1 - \Lambda^{(c)^2}}{1 + \Lambda^{(c)^2}} + (8N^2 - 5) Y \sin^2 \alpha \frac{3\Lambda^{(c)}}{2(1 + \Lambda^{(c)^2})} \right]. \tag{B15}$$

Proportional to Y^2, the second term in $\mu_{1,2}^{(T)}$ is small in comparison with the $(5/2 - N^2)$ term. It also does not affect the difference between $\mu_1^{(T)}$ and $\mu_2^{(T)}$, and, therefore, can be ignored. Using the identities

$$\frac{\Lambda}{1 + \Lambda^2} = \frac{1}{2\sqrt{g^2 + 1}}, \quad \frac{1 - \Lambda^2}{1 + \Lambda^2} = \frac{g}{\sqrt{g^2 + 1}}, \tag{B16}$$

and the expression (B6) for $g^{(c)}$ simplified by taking $X \ll 1$, yields solutions (31)–(33) for the refractive indices and eigenvectors of the electric field.

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