Plasma heating in stellarators at the fundamental ion cyclotron frequency

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The propagation and absorption of Alfvén waves at \( \omega \approx \omega_{ci} \) is studied in a straight nonuniform helical plasma with elliptical cross section, placed in an inhomogeneous \( l = 2 \) magnetic field created by helical windings. The model antenna with a spectrum of \( k_z \) induces the \( m = +1 \) poloidal harmonic. In a uniform plasma in a uniform magnetic field the antenna excites the \( m = +1 \) fast wave fields. These fields are right-handed almost everywhere in the plasma and they are not absorbed by plasma ions. In the nonuniform helical \( l = 2 \) magnetic field under certain conditions the antenna excites mostly the \( m = +1 \) fast wave and left-handed \( m = -1 \) slow wave fields. The coupling between these waves in the nonuniform magnetic field leads to the excitation of the left-handed wave field component near the plasma center and to the heating of plasma ions by this wave component. Collisionless absorption by plasma ions is assumed. The power deposition distribution in the plasma is examined for different plasma parameters. Results demonstrate that effective heating of the central part of the plasma at \( \omega = \omega_{ci} \) is possible for the low density plasmas in small scale devices. Because of the restrictions in the model, no firm conclusions can be made about the effectiveness of this method for high density plasmas in large scale devices. © 2000 American Institute of Physics. [S1070-664X(00)01002-8]

I. INTRODUCTION

Experiments on the L-2 Stellarator\(^1\) and GAMMA-10\(^2\) have demonstrated that fast Alfvén waves are efficiently absorbed at the fundamental ion cyclotron frequency in stellarators and mirror machines, and lead to central heating. This is due to coupling between fast (launched, \( m = +1 \)) and slow (absorbed, \( m = -1 \)) modes (\( m \) is the poloidal wave number) which are coupled through the inhomogeneity of the \( l = 2 \) fields. The coupling efficiency between these waves was investigated in Ref. 3. In this analysis a perturbation method was developed to find the structure of Alfvén wave modes in a cylindrical waveguide filled with a cold, collisional, uniform plasma with a vacuum layer between the plasma and a conducting wall when the magnetic field is a superposition of a uniform and an inhomogeneous \( l = 2 \) field created by helical windings. The influence of the helical field on the wave mode structure was treated as a perturbation. The result of this investigation demonstrated that coupling between fast and slow waves has a resonant behavior with the maximum coupling when a certain relation between the fast and slow wave wave numbers and the period of the winding is satisfied, so the coupling can be very effective, and may lead to effective plasma heating. This result is briefly discussed in Sec. III. The model used in this analysis, however, was very restrictive, so that no direct estimations of plasma heating efficiency could be made.

In the present analysis we estimate the heating efficiency of a stellarator plasma near \( \omega = \omega_{ci} \). We consider propagation and damping of Alfvén waves excited by a model antenna in a nonuniform cylindrical plasma with elliptical cross section, placed in an inhomogeneous \( l = 2 \) magnetic field created by helical windings. At \( \omega \approx \omega_{ci} \) in a uniform magnetic field, the \( m = +1 \) fast wave is a propagating wave provided the vacuum layer between the plasma edge and a conducting wall is of sufficient thickness,\(^3\) while for a broad range of plasma parameters the \( m = 0 \) and \( m = -1 \) fast waves do not propagate. This wave is right-handed and does not experience the ion cyclotron resonance. We restrict our consideration of the plasma heating scenario to the case where the antenna launches directly only the \( m = +1 \) fast wave fields, and the heating is due to the coupling of this wave to different slow wave modes, predominantly to the \( m = -1 \) slow wave. The model antenna induces the \( m = +1 \) poloidal harmonic and its \( k_z \) spectrum covers only fast wave modes. The excitation of slow wave fields by an antenna in a uniform magnetic field leads to the heating of the plasma surface. In a highly nonuniform stellarator magnetic field, the power deposition distribution by the directly excited slow waves may be different. The examination of this case is beyond the scope of our consideration. Antennas selectively exciting fast waves are realizable in practice, however more simple antennas with a broad \( k_z \) spectrum may be effective as well.

We consider collisionless wave damping by plasma ions in a simple model. The power deposition distribution in the plasma is examined for different plasma parameters. Because of the slow convergence of the numerical solutions at high magnetic field nonuniformities, which is due to the excitation of very high poloidal harmonics, the analysis is made for relatively small magnetic field nonuniformities. Results demonstrate that effective heating of the central part of the plasma near \( \omega = \omega_{ci} \) is possible for low density plasmas in small scale devices. Because of the restrictions due to the numerical method and in the model, no firm conclusions can be made about the effectiveness of this method for the high density plasma in a large scale device.

The results of our numerical analysis are in qualitative...
agreement with the results of our previous analysis of this problem using a simpler model.5

Section II contains a detailed description of the model used and approximations made in the analysis. It also includes a brief outline of the numerical method used to solve the equations derived in this section. In Sec. III the results of the analysis are discussed.

II. DESCRIPTION OF THE MODEL AND FIELD EQUATIONS

We consider a cylindrical geometry. The magnetic field is created by helical windings with the winding period L and the helicity \( \alpha = 2 \pi/l \). We approximate this magnetic field by an \( l = 2 \) expansion.8 The magnetic surfaces of such a field are concentric ellipses rotating with the frequency \( \alpha \) in the \( z \) direction. We suppose that the perfectly conducting boundary of the vacuum vessel lies on a magnetic surface. Let the semimajor and semiminor axes of the boundary ellipse be \( a \) and \( b \), respectively. We introduce the dimensionless helicity \( \epsilon = \alpha a \) and assume that length is measured in the units of \( a \). Then the components of the \( l = 2 \) magnetic field in cylindrical coordinates are

\[
\begin{align*}
\frac{B_r}{B_0} &= \mu \epsilon r \sin(2(\theta - \epsilon z)), \\
\frac{B_\theta}{B_0} &= \mu \epsilon r \cos(2(\theta - \epsilon z)), \\
\frac{B_z}{B_0} &= 1 - \mu \epsilon^2 r^2 \cos(2(\theta - \epsilon z)),
\end{align*}
\]

where \( B_0 \) is the magnetic field at \( r = 0 \). The last term in the equation for \( B_z \) is proportional to \( r^2 \) and it slightly changes the magnetic surface shape (some deviation from the elliptical) near the vessel boundary for the parameters used in our analysis, but this term is important for the approximation of the absolute value of the magnetic field in a stellarator configuration. The dimensionless parameters, \( \mu \) and \( \epsilon \), in this expansion influence the magnetic field nonuniformity, such that \( \mu \) is related to the ellipticity of the magnetic surfaces through the relation

\[
\frac{a}{b} = \left( \frac{1 + \mu}{1 - \mu} \right)^{1/2}.
\]

Let \( \rho \) be the semimajor axis of an ellipse corresponding to a magnetic surface. Then the equation of this surface is

\[
r^2 [1 - \mu \cos 2(\theta - \epsilon z)] = \rho^2 (1 - \mu).
\]

We assume that the plasma density is a function of \( \rho \) only. This model approximates a stellarator configuration.

We neglect finite Larmor radius effects because the condition \( (2 \pi \rho_{Li}/a)^2 \ll 1 \) is well satisfied for the parameters of our consideration and we perform the analysis only when the fields with low poloidal and radial wave numbers are excited.

Due to the rotational–translational symmetry of the problem, we introduce new coordinates \( z = z, \ r = r, \ \theta_1 = \theta - \epsilon z \). The electromagnetic fields in the waveguide are repeated periodically in the \( z \) direction with a period of \( 2L_0 \) and we assume that this length contains an integer number of winding periods. The wave fields decay with distance away from the antenna and \( L_0 \) is chosen to be large enough to ensure that the fields at this distance are sufficiently smaller than those near the antenna that the fields beyond \( L_0 \) may be neglected. The wave fields may then be expanded in a Fourier series, such that

\[
E(z, r, \theta_1) = \sum_{n = -\infty}^{\infty} E_n(r, \theta_1) \exp(ik_n z), \quad k_n = \frac{n \pi}{L_0}.
\]

A simple antenna induces \( m = +1 \) fields by the excitation of an electric field with time dependence \( \exp(-i \omega t) \) on the inner surface of the vessel and perpendicular to the \( z \) direction so that

\[
E^e = e_r e^{i m \theta} \exp(-z^2/\zeta_A^2),
\]

with \( m = +1 \) and where the components of the vector \( e_r \) are everywhere parallel to the surface, and in a cylindrical basis depend upon \( \theta_1 \) only. The Fourier components of this antenna field are

\[
E^e_n = e_r \frac{\sqrt{\pi \zeta_A}}{2L_0} e^{i \theta_1} \exp\left[ -\frac{1}{4} (\epsilon - k_n)^2 \zeta_A^2 \right].
\]

The antenna width \( \zeta_A \) is chosen to be relatively wide so that the antenna spectrum covers primarily fast wave fields and almost no direct excitation of the \( m = +1 \) slow waves takes place.

In a magnetic field with a relatively small nonuniformity, such an antenna excites mostly \( m = +1 \) fast and \( m = -1 \) slow wave fields, so that in \( z, \ r, \ \theta \) coordinates the fields excited by the antenna for a fixed \( k_n \) are approximately of the form:

\[
E(z, r, \theta_1) = E_j(r) \exp[i \theta + i(k_n - \epsilon) z] + E_s(r) \exp[-i \theta + i(k_n + \epsilon) z],
\]

where the first term is a fast wave field and the second is a slow wave field. The fast wave fields in Eq. (3) are right-handed in the bulk of the plasma for all \( k_n \) in the antenna spectrum of Eq. (2). These fields are well described by the Maxwell equations with a local cold plasma dielectric tensor. In the limit when magnetic field has only a \( z \) component, the excitation of the slow wave fields in Eq. (3) is described with a hot plasma dielectric tensor (see, for example, Ref. 7) with the dimensionless \( k_1 = k_n + \epsilon \). In the limit when \( \rho_{Li} \rightarrow 0 \) this dielectric tensor is local with respect to the \( r, \ \theta_1 \) coordinates and has the form

\[
K = \begin{pmatrix} K_1 & K_2 & 0 \\ -K_2 & K_1 & 0 \\ 0 & 0 & K_3 \end{pmatrix},
\]

where for a single ion species plasma,
and $Z$ is the plasma dispersion function. Because the fast wave fields do not depend on the resonant ion terms in this tensor, it appropriately describes their excitation as well. We apply this dielectric tensor for the investigation of wave excitation and absorption in a stellarator magnetic field with relatively small nonuniformity. From Eq. (1) it follows that this approximation is valid when $\mu \epsilon r \ll 1$. This condition is accurate near the plasma center but not very accurate near the plasma periphery. The plasma density, however, which decays toward the periphery, suppresses the inaccuracy in this description. We assume a uniform temperature distribution in the plasma and find the power absorbed per unit volume from $\langle J \cdot E \rangle$.

Introducing the dimensionless parameter $\lambda = \omega a / c$, then for a fixed $k_n$, the Maxwell equations for the $n$th Fourier component of the electric fields are

\[
K_1 = 1 - \frac{\omega_{pe}^2}{\omega^2 - \omega_{ci}^2} + \frac{1}{2} \frac{\omega_{pi}^2}{\omega^2} \frac{Z}{|V_i|} (\omega - \omega_{ci}), \quad K_2 = \frac{i \omega_{pe} \omega_{ci}}{\omega (\omega^2 - \omega_{ci}^2)} + \frac{i}{2} \frac{\omega_{pi}^2}{\omega} (\omega - \omega_{ci}), \quad K_3 = 1 - \frac{\omega_{pe}^2}{\omega^2} + \frac{1}{2} \frac{\omega_{pi}^2}{\omega^2} \frac{Z}{|V_i|} (\omega - \omega_{ci}),
\]

where dimensionless $\nu_i$ is

\[
\nu_i = \frac{1}{\omega a} \frac{2 \kappa T_i}{m_i},
\]

and $Z$ is the plasma dispersion function. Because the fast wave fields do not depend on the resonant ion terms in this tensor, it appropriately describes their excitation as well. We apply this dielectric tensor for the investigation of wave excitation and absorption in a stellarator magnetic field with relatively small nonuniformity. From Eq. (1) it follows that this approximation is valid when $\mu \epsilon r \ll 1$. This condition is accurate near the plasma center but not very accurate near the plasma periphery. The plasma density, however, which decays toward the periphery, suppresses the inaccuracy in this description. We assume a uniform temperature distribution in the plasma and find the power absorbed per unit volume from $\langle J \cdot E \rangle$.

Introducing the dimensionless parameter $\lambda = \omega a / c$, then for a fixed $k_n$, the Maxwell equations for the $n$th Fourier component of the electric fields are

\[
\left( \frac{1}{r^2} + \epsilon \right) \frac{\partial^2 E_r}{\partial \theta_1^2} - 2ik_n \epsilon \frac{\partial^2 E_r}{\partial \theta_1^2} - k_n^2 E_r - \frac{1}{r} \frac{\partial^2 E_\theta}{\partial \theta_1^2} - \frac{1}{r^2} \frac{\partial E_\theta}{\partial \theta_1} + \frac{1}{r^2} \frac{\partial E_\theta}{\partial \theta_1} - \frac{1}{r^2} \frac{\partial E_\theta}{\partial \theta_1}
\]

\[+ \epsilon \frac{\partial^2 E_r}{\partial r \partial \theta_1} - ik_n \epsilon \frac{\partial E_r}{\partial \theta_1} = -\lambda^2 (TE)_r,
\]

\[\frac{1}{r^2} \frac{\partial^2 E_r}{\partial \theta_1^2} + \frac{1}{r \partial \theta_1} \frac{\partial E_r}{\partial \theta_1} - \frac{1}{r^2} \frac{\partial^2 E_\theta}{\partial \theta_1^2} + \frac{1}{r} \frac{\partial^2 E_\theta}{\partial \theta_1^2} + \frac{1}{r^2} \frac{\partial E_\theta}{\partial \theta_1} - \frac{1}{r^2} \frac{\partial E_\theta}{\partial \theta_1}
\]

\[\frac{1}{r} \frac{\partial^2 E_r}{\partial \theta_1^2} - \frac{1}{r} \frac{\partial^2 E_r}{\partial \theta_1^2} + \epsilon \frac{\partial^2 E_\theta}{\partial \theta_1^2} - \frac{1}{r} \frac{\partial E_\theta}{\partial \theta_1} - \frac{1}{r} \frac{\partial E_\theta}{\partial \theta_1} - \frac{1}{r} \frac{\partial E_\theta}{\partial \theta_1}
\]

\[= -\lambda^2 (TE)_\theta,
\]

\[\frac{1}{r} \frac{\partial^2 E_r}{\partial \theta_1^2} - ik_n \epsilon \frac{\partial (r E_r)}{\partial \theta_1} - \frac{1}{r} \frac{\partial^2 E_\theta}{\partial \theta_1^2} + \frac{1}{r} \frac{\partial E_\theta}{\partial \theta_1} + \frac{1}{r} \frac{\partial E_\theta}{\partial \theta_1}
\]

\[+ \frac{1}{r^2} \frac{\partial^2 E_\theta}{\partial \theta_1^2} + \frac{1}{r^2} \frac{\partial E_\theta}{\partial \theta_1} = -\lambda^2 (TE)_z.
\]

The tensor $\mathbf{T}$ is obtained from the dielectric tensor of Eqs. (4) and (5) by the tensor transformation rule from the basis connected with the local direction of the magnetic field to the cylindrical basis. The boundary conditions on the vessel surface are the matching of the tangential components of the electric field with the antenna field of Eq. (2). The equations (6) are solved numerically by a finite difference method. For this the change of variables $r \rightarrow \rho$ of the form

\[
\rho = \frac{r}{\sqrt{1 - \mu \cos^2 \theta_1}}
\]

is made and the transition to the basis corresponding to the coordinate lines of $\rho$, $\theta_1$, $z$ is made as well. For the consistency of the difference method, the condition $\mathbf{V} \cdot \mathbf{E} = 0$ is enforced by the elimination of the radial field component from the difference equations with the help of this condition.

\[\text{III. RESULTS AND DISCUSSION}
\]

For our analysis, we consider a vessel with semimajor axis $a = 17$ cm and fixed helicity of the windings, $\epsilon = 0.9$. This helicity is within the range of helicities used in stellarators. The magnetic field on the axis is $B_0 = 0.1 T$, corresponding to $f_c = 1.5 MHz$ for an hydrogen plasma, and the ion temperature is $T_i = 10 eV$. The plasma density profile is of the form $n = n_0 \rho (\rho)$, with $n(\rho)$ shown in Fig. 1. We keep a finite plasma density (1% of $n_0$) near the vessel surface in order to suppress the excitation of surface waves on the boundary between plasma and vacuum. The surface waves near the vessel boundary, however, are excited. As long as the plasma density near the vessel surface is low, its profile does not have much influence on the wave fields in the bulk of the plasma. In our calculations we keep $\mu \leq 0.5$, which corresponds to a magnetic field with relatively small nonuniformity.

Figure 2 shows the dispersion curves for different waves in a simple model when a uniform plasma is placed in a uniform magnetic field in a cylinder with circular cross section. These curves are obtained from a cold plasma model. The dispersion curves for the slow waves with different poloidal numbers $m$ are close to those presented in Fig. 2 when the frequency is near $\omega_{ci}$. In the previous analysis, it was demonstrated that the effective coupling between the excited fast wave and a slow wave occurs when the condition

\[
k_s = k_f + \omega (m_f - m_s) / \alpha
\]

is satisfied. In this equation $k_f$ is a slow wave number and $k_s$ is the wave number of the excited fast wave. In our analysis $m_f = +1$ and $m_s = -1$. One should expect effective heating of the bulk of the plasma when condition (7) is approximately satisfied near the plasma center.

We analyze the wave excitation and absorption within
we investigate the excitation and absorption of Alfvén waves by the antenna. In a uniform magnetic field when either antenna spectrum covers the range of the heating. For every plasma parameter under investigation, the model at \( \omega = \omega_{ci}(\rho = 0) \). Under this condition the resonance lines \( (\omega = \omega_{ci}) \) cross on the axis which enables central heating. For every plasma parameter under investigation, the antenna excites fast wave fields which are right-handed in the bulk of the plasma and are not absorbed by the plasma ions.

In the nonuniform magnetic field with \( \mu = 0.5, \epsilon = 0.9 \), we investigate the excitation and absorption of Alfvén waves for the range of plasma densities between \( 0.5 \times 10^{11} \) and \( 1.6 \times 10^{12} \text{ cm}^{-3} \). In this nonuniform magnetic field there is a substantial amount of the left-handed field component in the plasma center, and a significant amount of the wave energy is absorbed there. Figure 3 shows the power \( P(\rho) \) dissipated per unit volume averaged over the magnetic surface \( L_0 = z \leq L_0 \) versus the position \( \rho \) of the surface for different plasma densities. Each plot is normalized to its own \( P(0) \). The excitation of strong surface fields near the antenna and our assumption that there is a finite plasma density in this region lead to the artificially enhanced localized absorption of the wave fields near the antenna. The proper modeling of the near antenna effects is beyond the scope of our analysis and this region is not presented in our results. The results of Fig. 3 demonstrate that the averaged power deposition has a maximum in the center for low plasma densities and that the power deposition on the plasma periphery increases with the increase of the density. This monotonic change of the power deposition distribution with the increase of the plasma density is in agreement with condition (7). At low densities at \( \omega = \omega_{ci} \) the last term in Eq. (7) covers "most of the distance" between fast and slow wave curves (see Fig. 2), so that condition (7) is approximately satisfied. For higher densities the distance between curves increases while the last term in Eq. (7) is unchanged, so that the coupling between waves becomes ineffective and the relative amount of power deposited in the plasma center decreases.

Figures 4a–4d show the distribution of the power absorbed per unit volume (arbitrary units) in the plane of the antenna \( (z = 0) \) for four different densities for the same parameters as in Fig. 3 except \( \mu = 0.45 \). The power is absorbed mostly along one of the resonant lines \( (\omega = \omega_{ci}) \) and the maximum of the absorption lies at some distance from the plasma center. After averaging over magnetic surfaces, the maximum of the averaged absorbed power moves toward the plasma center and results in central heating of the plasma. At higher densities the maximum of the absorbed power moves toward the periphery and the relative power dissipated in the center decreases.

The lack of up–down and left–right symmetry in Figs. 4a–4d is an unexpected result and leads to a somewhat surprising conclusion. The left-handed \( m = -1 \) slow wave, generated through the coupling resonance, mixes with a weak left-handed component of the \( m = +1 \) fast wave in such a way as to enhance the field along one resonance line as shown. From the perturbation analysis of this problem, the electric field has a number of components, given there by Eq. (68):

\[
E(z, r, \theta) = \sum_{m \neq m_0} E_m(r) e^{i(m_0 \theta + m \theta)} + \sum_{m = m_0} E_{m_0}(r) e^{i(m_0 \theta + m_0 \theta)} e^{ik_0 \Delta z},
\]

where for our case, \( m_0 = \pm 1 \). The \( E_0(r) \) term is the field in a uniform plasma with a vacuum layer in a cylindrical waveguide without the helical perturbation. This term is primarily right-handed, but has a left-handed component near the plasma edge that vanishes as \( \omega \rightarrow \omega_{ci} \). The \( E_{m_0}(r) \) term, on the other hand, is due to the inhomogeneity from the perturbing magnetic field. This field component also has a small left-handed component which is largest near the plasma edge and decays toward the plasma center, but this left-handed component does not vanish as \( \omega \rightarrow \omega_{ci} \). It is the left-handed component of this term that mixes with the left-handed \( m = -1 \) slow wave induced by the coupling resonance to give
the observed asymmetry. Analysis of the wave fields in the cases shown clearly indicates that there is an $m=1$ left-handed component varying even on the cyclotron resonance lines with phase and amplitude to minimize heating on one line and maximize the heating on the other line. This mixing vanishes as $r\rightarrow 0$, so that near the axis, the heating is symmetric. If the plasma density and/or the plasma size is increased enough, then the $m=0$ and $m=-1$ fast waves can also be excited, and the plasma heating then becomes more nearly symmetric.

In order to analyze the transition from the uniform magnetic field when only right-handed fast wave fields are excited to the nonuniform $l=2$ magnetic field when left-handed slow wave fields are induced due to the coupling between these modes, we investigate the power deposition distribution for fixed $n_0$, $\epsilon$ and antenna spectrum while varying $\mu$. Figure 5 shows the average power $P(\rho)$ (defined in Fig. 3) versus $\rho$ for increasing $\mu$, with $n_0=10^{11}\text{ cm}^{-3}$ and $\epsilon=0.9$. All plots in Fig. 5 are normalized to $P(0)$ for $\mu=0.5$. This figure demonstrates the changes to the averaged power deposition distribution and to the magnitude of the absorbed power with the increase of the $l=2$ magnetic field nonuniformity parameter $\mu$. In the uniform magnetic field with $\mu=0$ there is no absorption in the bulk of the plasma and some absorption near the plasma edge. With the increase of $\mu$, the amount of the left-handed field component in the plasma center increases, resulting in the gradual increase of the power dissipated there and contributes to the central heating of the plasma.

With the change of the magnetic field strength $B_0$, the dispersion curves in Fig. 2 experience only slight changes. This result, when combined with condition (7), lead to the conclusion that the heating efficiency should not strongly depend on the value of $B_0$ provided $\omega/\omega_i\approx 1$ near the center. With the increase of the plasma density, $n_0$, and of the size of the device, $a$, the absolute distance between the plots of the fast and slow wave modes increases while the helicity $\epsilon$ is unchanged, resulting in the diminishing of the central heating.
plasma heating within our model. The magnetic field nonuniformity in real stellarators is significantly stronger than in our model, however, and this should improve the centrality of the heating for higher plasma densities. For larger \( \mu \) and \( \epsilon \), very high poloidal and radial modes are excited such that our code cannot resolve these small scale fields. Due to this fact and because our model is not accurate for the higher nonuniformities of the magnetic field, we are unable to estimate the effectiveness of this method for high density plasmas in large scale devices.

**IV. CONCLUSIONS**

While the conclusions from this investigation are restricted due to numerical limitations, the likelihood that this scheme of heating stellarators at the fundamental will be important is still high enough that it merits further work. The resonant behavior for fast–slow Alfvén wave coupling that was predicted by the original perturbation analysis\(^3\) is apparently confirmed in these calculations. Its interpretation is modified somewhat, however, since in the perturbation analysis, the matching condition was a global condition where both of the parallel wave numbers were constant across the entire waveguide. From the results of this investigation, however, it appears that the condition is more nearly a local condition, since both of the wave numbers vary over the cross section, and the slow wave only propagates in the upper and lower quadrants of Fig. 4. As a local condition, we may write Eq. (7) approximately as

\[
k_s(r, \theta) = k_f(r, \theta) + \epsilon (m_f - m_s),
\]

where

\[
k_{s,f} \approx \frac{\omega_p}{c} \left( \frac{n(r)}{\Omega(r, \theta)} \right)^{1/2}
\]

and \( \Omega(r, \theta) = \omega_c(r, \theta)/\omega \). For the fast wave expression, there should be an approximate transverse wave number included, but for the \( m = +1 \) fast wave mode, \( k_s \) is very small (there is virtually no cutoff for this mode with a large vacuum layer), and so we neglect it. If the \( m = 0 \) and \( m = -1 \) fast wave modes propagate, then an appropriate value for \( k_s \) should be included for their matching condition. The location of this local matching condition is shown in Fig. 6 for the parameters of Figs. 4(a)–4(d), showing that the coupling region between the fast wave and the slow wave moves outward as the density increases. While this helps us to understand why the heating moves outward with increasing density, it is not very useful in determining at what radius the maximum heating will occur, since the coupling resonance surfaces correspond to \( |\xi| = |(\omega - \omega_c)/k_s V_t| < 1 \) in all cases so that it is in a strong absorption region. This means that the coupling resonance is not sharp since \( k_s \) has a substantial imaginary part and it means that the slow wave is strongly damped so that the absorption is localized near the plane of the antenna.

There is one other effect mentioned in relation to Eq. (8) that also modifies the interpretation of the local coupling resonance as being a place where the fast wave couples to the slow wave and therefore leads to heating. Numerical analysis of the wave fields show that the fast wave has a small left-handed component even at the resonance so that heating may occur without any coupling to the slow wave at all. The relative amounts of absorption from these two distinct sources is resolved through the calculation of the wave field components which are \( m = -1 \) for the slow wave and \( m = +1 \) for the fast wave component. Although the \( m = -1 \) slow-wave component dominates, this other component is observed.

One possible way to improve the coupling and the centrality of the heating for higher densities is to reduce the frequency \( \omega \) [see Eq. (7) and Fig. 2]. This leads to a shifting of the resonance from the plasma center, but if the magnetic field nonuniformity is strong enough and the temperature is high enough, the position of the strongest absorption will still be close enough to the plasma center that efficient plasma heating may be realized near the plasma center. Because of numerical convergence difficulties, we were not able to verify this hypothesis, but it seems likely in view of the general results obtained.

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7. D. G. Swanson, Plasma Waves (Academic, Boston, 1989), Chap. 4.3.