Frequency spectra at large wavenumbers in two-dimensional Hasegawa-Wakatani turbulence

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October 30th, 2012
2012 APS-DPP Conference, Providence, RI
In the Hasegawa-Wakatani model, the frequency spectra have shown the following.

- In the adiabatic regime where vorticity advection is dominant,
  - the nonlinear frequency $\tilde{\omega}(k)$ for each wavenumber increases with $k_y$ and is shifted in the electron drift direction from linear eigen-frequency of the linearly unstable modes $\omega_\ell$.
  - This spectrum shows that the spectral width $\Delta \tilde{\omega}$ is smaller than the nonlinear frequency $\tilde{\omega}$ even at large wavenumbers.
  - The change of the cross-phase, $\Delta \theta^{\phi \psi^*}(k, \omega)$, between potential and density fluctuations are negatively correlated with the nonlinear frequency shift $\delta \omega = \omega - \omega_\ell(k)$.

- In the hydrodynamic regime where density advection is strong, broad spectrum with almost negligible nonlinear frequencies, $\Delta \tilde{\omega} > \tilde{\omega}$ and random cross-phase $\Delta \theta^{\phi \psi^*}(k, \omega) \simeq 0$.

Emerging nonlinear frequency in both cases can be explained by three-wave nonlinear interaction with combination of vorticity and density advection, and the change of the cross-phase in frequency can be explained by the relation between frequency and phase via linear drift waves including linearly stable response.
Overview

- Introduction
  - Motivation
  - Introduction to the Hasegawa-Wakatani model

- Interesting features of frequency spectra
  - Nonlinear frequencies and frequency broadening
  - Nonlinear cross-phase between potential and electron density

- Analysis of Frequency Spectrum
  - Cross-phase dependence on nonlinear frequency
  - Nonlinear frequency shift by vorticity advection
  - Role of Density advection

- Conclusion and Discussion
Motivation: frequency vs. wavenumber spectra

- Frequency spectra $P(\omega)$ of turbulent fluctuations are easy to measure in experiments, while wavenumber spectra $P(k)$ are relatively easy to obtain in theory and computation. Therefore, better understanding of the $\omega - k$ spectrum of fluctuations is helpful to bridge them.

- The mapping between wave number spectrum and frequency spectrum are considered to be quasilinear, $\omega = \omega_{\ell}(k)$ where $\omega_{\ell}$ is a linear eigenmode frequency, for micro-instabilities at energetically dominant wavenumbers. However, the relations can be intractably complicated.

- There have been recent researches to explore nonlinear effect on the frequency spectrum such as energy transfer among the fluctuations of different frequencies, and the non-symmetric frequency shapes.

- While these efforts clearly demonstrates a certain relation between wavenumber spectrum vs. frequency spectrum, mostly either spectrum is specified as frequency spectrum $P(\omega, x_0)$ at a fixed spatial location or wavenumber spectrum $P(k, t_0)$ in time basis.
Motivation: nonlinear energy transfer

- A frequency spectrum at a large wavenumber $k$ is expected to be broad since fluctuations is highly nonlinear at those wavenumbers. There should be strong nonlinear transfer to the wavenumbers where energy damping is strong enough to balance total energy injection into the system of fluctuations.

- Kim and Terry (2011) has shown that three-wave coupling with complex frequency, i.e. including linear growth rates, from generalized one-fluid model can have the equilibrium that satisfies energy balance. That implies that finite nonlinear frequency different from linear frequencies can represent consistent energy transfer which can not be explained by turbulent diffusion, that is, frequency broadening.

- Can nonlinear frequencies be identified in more general models, i.e. two(or more)-fluid or gyrokinetic model? and are those frequencies related to nonlinear energy transfer?

Therefore, we investigate $\omega - k$ spectrum $P(\omega, k)$ and nonlinear frequencies in the Hasegawa-Wakatani model where the nature of turbulence can change?
The HW model (Hasegawa and Wakatani 1983) is well-known to describe the collisional drift wave turbulence. The HW turbulence covers from weak ($\tilde{\omega} \gg \Delta \tilde{\omega}$) to strong ($\tilde{\omega} \ll \Delta \tilde{\omega}$) turbulence. The frequency spectrum has peaked around $\omega \sim \omega_*$, it is also observed to be broad.

Gang, Diamond et al. 1991 constructed correlation functions via the EDQNM closure and showed that in adiabatic regime, cross-correlation between density and potential inhibits the density advection and in hydrodynamic regime, the depression of cross-correlation leads to the coherent structure where the density fluctuation is trapped within vorticity. Only the frequency broadening with the renormalization $\Delta \tilde{\omega} \sim \Theta(\Delta \tilde{\omega}) |\phi_k|^2$ is included while keeping $\tilde{\omega} \sim \omega_\ell$.

Hu, Krommes et al. 1997 describes HW turbulence in the EDQNM systemically derived from Direction Interaction Approximation(Kraichnan 58) and shows the smooth transition from weak $E \sim \gamma/\omega$ to strong turbulence $\sim (\gamma/\omega)^2$ in terms of the adiabaticity parameter. While DIA closure does not exclude the deviation of nonlinear frequency from the linear frequency, it is not yet clear.

Our focus is the description of the nonlinear frequency $\tilde{\omega}$ and $\Delta \tilde{\omega}$ in the hydrodynamic and adiabatic regime.
the Hasegawa-Wakatani model

In the wave number space, the 2D homogeneous HW model is

\[
\frac{\partial \phi}{\partial t} = -\frac{\alpha}{k^2} (\phi - \psi) - \nu k^2 + \sum_{\textbf{k}+\textbf{k}'+\textbf{k}''=0} \hat{z} \cdot \textbf{k}' \times \textbf{k}'' \left( \frac{k''^2 - k'^2}{k^2} \right) \phi'^* \phi''^*
\]

\[
\frac{\partial \psi}{\partial t} = \alpha (\phi - \psi) - i k_y \kappa \phi - \mu k^2 \psi + \sum_{\textbf{k}+\textbf{k}'+\textbf{k}''=0} \hat{z} \cdot \textbf{k}' \times \textbf{k}'' \left( \phi'^* \psi''^* - \phi''^* \psi'^* \right).
\]

where \( \phi \) and \( \psi \) are electrostatic potential and electron density fluctuations.

- The adiabatic parameter \( \alpha \) characterizes the degree to which electrons can move rapidly along the magnetic field lines and establishes a perturbed Boltzmann density response, where the density advection is zero.

- The HW model is well-known to exhibit both an adiabatic regime \((\alpha \gg 1)\) and a hydrodynamic regime \((\alpha \ll 1)\).

- In the model, the phase between \( \phi \) and \( \psi \) evolves dynamically and self-consistently, compared to one-field Terry-Horton(TH) mode with the fixed cross-phase \((i\delta \text{ model})\). In the adiabatic limit, \( \alpha \to \infty, \phi \to \psi \) and the HW model reduces to a Hasegawa-Mima equation.
The complex linear eigen-frequencies $\omega$ are

$$
\omega_{1,2} = -\frac{i}{2} \left[ \alpha \left( 1 + \frac{1}{k^2} \right) + (\nu + \mu)k^2 \right] 
\pm \frac{i}{2} \left\{ \left[ \alpha \left( 1 - \frac{1}{k^2} \right) - (\nu - \mu)k^2 \right]^2 + \frac{4\alpha}{k^2} (\alpha - i\omega^*) \right\}^{1/2}.
$$

There are two eigenmode branches, unstable $\omega_1$ and stable $\omega_2$. Given the linear eigen-frequency $\omega_\ell$, the ratio between two fields are

$$
\frac{\phi_\ell}{\psi_\ell} = \frac{1 - i\omega_\ell/\alpha + \mu k^2/\alpha}{1 - i\omega^*/\alpha} = \frac{1}{1 - i\omega_\ell k^2/\alpha + \nu k^4/\alpha}.
$$

In the adiabatic regime where $\alpha \gg \omega_\ell, \omega^*, \mu k^2$ and $\nu k^2$, the complex ratio becomes

$$
\frac{\phi_\ell}{\psi_\ell} \approx 1 - i \frac{\omega_\ell - \omega^*}{\alpha} \approx 1 + i \frac{\omega_\ell k^2}{\alpha}.
$$

Combining the ratio with $\Gamma \sim \text{Re} \langle ik_y \phi^* \psi \rangle$ provides that the necessary condition for the instability is

$$
0 < \omega_\ell < \omega^*
$$
for a wavenumber $k_y > 0$. 
The quadratic equation can be constructed.

\[
\frac{dE}{dt} = \Gamma - D_\parallel - D_\phi - D_\psi 
\]

\[
\frac{d}{dt}\langle \psi\omega \rangle = \alpha \left[ \psi(\phi - \omega) - |\nabla \phi|^2 - |\psi|^2 \right] - (\mu + \nu) \langle \nabla \psi \cdot \nabla \omega \rangle ,
\]

where

\[
E = \left\langle \frac{|\psi|^2 + |\nabla \phi|^2}{2} \right\rangle, \quad \Gamma = \langle \psi v_x \rangle = \left\langle -\psi \frac{\partial \phi}{\partial y} \right\rangle
\]

\[
D_\parallel = \left\langle \alpha |\phi - \psi|^2 \right\rangle, \quad D_\phi = \left\langle \nu |\omega|^2 \right\rangle, \quad D_\psi = \left\langle \mu |\nabla \psi|^2 \right\rangle
\]

- \( D_\parallel \) depends on \(|\phi - \psi|\). In the adiabatic regime, the main dissipation is \( D_\phi \) and \( D_\psi \), while in the hydrodynamic regime, \( D_\parallel \) may be relevant.

- Also, The evolution of \((\psi - \omega)^2\) is also dependent on the quadratic terms.(Hasegawa and Wakatani 1983).
Two-dimension HW simulation

- The simulation data will show
  - Detailed frequency spectra ($\alpha, k_y$),
  - Time scale comparison, and
  - Amplitude fluctuations.

- Full simulation with the resolution of $256^2$.

- With the fixed $\kappa = 1, \nu = 0.01, \mu = 0.01, \Delta k_x = \Delta k_y = 0.1$.

<table>
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- Each time series has $N = 8000$ and $\Delta t = 0.05$, so $\delta \omega = 0.0078$.

- Each moment of $\phi$ and $\psi$ is averaged over 15 ensembles (10 for ADI).
Good saturation with the nonlinear energy deviation $\sim 10^{-8}$

$\text{HYD} \rightarrow \text{ADI (arb. unit)}$

- $E_\phi : 3 \rightarrow 2$
- $E_\psi : 180 \rightarrow 4$
- $\frac{E_\phi}{E_\psi} : 0.016 \rightarrow 0.5$
- $\frac{\Gamma_n}{\sqrt{E_\phi E_\psi}} : 0.32 \rightarrow 0.12$
Potential and Density Fluctuations

**HYD** (potential, vorticity, density)

- $\phi(t = 1.950e+03)$
- $\omega(t = 1.950e+03)$
- $\psi(t = 1.950e+03)$

**qADI** (potential, vorticity, density)

- $\phi(t = 2.000e+03)$
- $\omega(t = 2.000e+03)$
- $\psi(t = 2.000e+03)$
Relevant Time Scales

- linear frequency and growth rate: $\omega_\ell$ and $\gamma_\ell$
  - diamagnetic frequency $\omega_* = k_y \kappa$. In $\alpha = 2.0$, $\omega_\ell \simeq \omega_*$ for the unstable branch $k_y < 0.3$.
  - parallel dissipation time scales: $\alpha$ or $\alpha/k^2$
  - perpendicular dissipation time scales: $\mu k^2$, $\nu k^2$; these are mostly insignificant.

- linear decorrelation time from wave packets $\omega_{ac,lin} = \Delta k \partial \omega_\ell / \partial k \ll \omega_\ell$

- nonlinear frequency $\tilde{\omega}$ and decorrelation time $\Delta \tilde{\omega}^{-1}$ where defined as with the assumption of Lorentzian distribution,

$$
\tilde{\omega} = \int P(\omega) \omega d\omega \quad \text{and} \quad \int_{\tilde{\omega} - \Delta \tilde{\omega}}^{\tilde{\omega} + \Delta \tilde{\omega}} P(\omega) d\omega = \frac{1}{2}
$$

with the normalized spectrum $P(\omega)$. 
In the weak turbulence theory, an observed correlation function takes the form
\[
\langle \phi^*_k(t') \phi_k(t) \rangle = I(\omega(\mathbf{k})) e^{-\omega(\mathbf{k})(t-t')} \rightarrow \langle \phi^*_k(\omega) \phi_k(\omega) \rangle = I(\omega(\mathbf{k})) \delta(\omega - \omega(\mathbf{k}))
\]
where a wave feature, \( \omega = \omega(\mathbf{k}) \), is observed. With a finite auto-correlation time \( \tau_{ac} = \Delta \tilde{\omega}_k^{-1} \), the spectrum can be extended to the Lorentzian form,
\[
P_{wt}^k(\omega) = |\phi_k(\omega)|^2 \propto \frac{\Delta \tilde{\omega}_k}{(\omega - \omega(\mathbf{k}))^2 + \Delta \tilde{\omega}_k^2}.
\]
where \( \Delta \tilde{\omega}_k \sim \gamma \ell \) since the nonlinear damping must balance the linear instability. This spectral form produces the auto-correlation function
\[
\langle \phi^*_k(t') \phi_k(t) \rangle \propto e^{-i\omega(\mathbf{k})(t-t') - \Delta \tilde{\omega}_k |t-t'|},
\]
This spectral form can be generalized by replacing linear frequency \( \omega(\mathbf{k}) \) with nonlinear frequency \( \tilde{\omega}_k \)
\[
P_k(\omega) = \frac{1}{\pi} \frac{\Delta \tilde{\omega}_k}{(\omega - \tilde{\omega}_k)^2 + \Delta \tilde{\omega}_k^2}.
\]
Strong turbulence does not feature a wave frequency \( \omega \sim \omega(\mathbf{k}) \) due to either the short auto-correlation time \( \omega(\mathbf{k}) \ll \Delta \omega_k \), or lack of the wave frequency. The fluctuations qualitatively become quasi-random arising from nonlinear interaction.
\[
P_{st}^k(\omega) = \frac{1}{\pi} \frac{\Delta \tilde{\omega}_k}{\omega^2 + \Delta \tilde{\omega}_k^2}.
\]
Nonlinear Frequency $\tilde{\omega}$ and spectral width $\Delta \tilde{\omega}$

\[ \omega(\phi_k): k_x = 0.000#12a \]

\[ \omega(\phi_k): k_x = 0.000#14a \]

- $\tilde{\omega} \ll \Delta \tilde{\omega}$ for $\phi$ and $\psi$ for most $k$ except extremely low $k$.
- $\tilde{\omega} \sim \omega_\ell$ for low $k$ and $\tilde{\omega}$ remains finite and at the order of linear scale $\omega_\ell$ and $\gamma_\ell$.
- $\Delta \tilde{\omega} \propto k_y$

Even at large $k$ where nonlinear energy transfer is supposed to be strong, nonlinear broadening does not take place.
Nonlinear Frequency $\tilde{\omega}$ and spectral width $\Delta\tilde{\omega}$

- For low $k_y < 1.0$: $\tilde{\omega}(\phi) \simeq \tilde{\omega}(\psi) \simeq \omega_\ell$ and for large $k_y > 1.5$, the nonlinear frequencies $\tilde{\omega}(\phi)$ increases in $k_y$ while $\tilde{\omega}(\psi)$ remains almost constant.

- The frequency spectral width $\Delta\tilde{\omega}$ increase linearly in $k_y$ for both fluctuations.

- At large $k$, nonlinear frequency and broadening can be modeled as a linear function of $k_y$.

- Nonlinear frequencies of potential and density fluctuations behave differently $\tilde{\omega}(\phi) \neq \tilde{\omega}(\psi)$.
Adiabaticity and a linear relation

The proportionality $d(\tilde{\omega}, \Delta\tilde{\omega})/k_y$ for large $k_y$ is compared over different $\alpha$s.

- As $\alpha$ increases, $\frac{d\tilde{\omega}}{dk_y}$ increases and $\frac{d\Delta\tilde{\omega}}{dk_y}$ decreases.

- Nonlinear frequency for potential fluctuation increases faster in $k_y$ than for density fluctuation.

$$\frac{d\tilde{\omega}}{dk_y}(\phi) > \frac{d\tilde{\omega}}{dk_y}(\psi).$$
For $\alpha$s, the phase difference $\arg(\phi\psi^*)$ is of linearly unstable relations for small $k_y$ and decreases for large $k_y$.

The phase difference tends to be zero in HYD implying that the phases are random: $\arg(\phi\psi^*) \to 0$.

The phase difference remains finite in ADI. And for the intermediate $\alpha$s, there is smooth transition between two extreme $\alpha$s.
The nonlinear frequency shift $\Delta \tilde{\omega}$ and the cross phase $\Delta \tilde{\theta}_{\phi \psi^*}$ are well correlated above $k_y \simeq 1.3$.

The positive shift in frequency and the negative shift in cross-phase shift are also observed for the individual wavenumber spectrum $\tilde{\omega}(k)$.

$$\delta \tilde{\omega}(k) \Delta \tilde{\theta}_{\phi \psi^*} < 0$$
“Energetic” point of view of a spectrum

- In a steady state (if it can), the fluctuation can be expressed in terms of Fourier series in frequency and wavenumber space.

\[ \phi(x, t) = \int \int dt dx e^{-i(\omega t - k \cdot x)} \Phi(k, \omega). \]

- When an equation

\[ \frac{\partial \phi}{\partial t} (t, x) = L \left( \frac{\partial}{\partial x} \right) \phi(t, x) + N \left( \frac{\partial}{\partial x} \right) \phi(t, x) \phi(t, x) \]

describes turbulence, then taking the Fourier series of it and multiplying with the conjugate leads

\[ -i\omega |\Phi(\omega, k)|^2 = L(k) |\Phi(\omega, k)|^2 + \hat{N}(\omega, k) \]

where

\[ \hat{N}(\omega, k) = \int N(k, k') \Phi^*(\omega, k) \Phi^*(\omega', k') \Phi^*(-\omega' - \omega, -k' - k). \]

- Then the quadratic equation (or energy equation) is

\[ 0 = \Re L(k) |\Phi(\omega, k)|^2 + \Re \hat{N}(\omega, k) \quad \omega |\Phi|^2 = -\Im L(k) |\Phi|^2 - \Im \hat{N} \quad \text{energy balance} \]
(Left) : two three-wave interactions contributing dominantly to \( k = (0.0, 1.0) \),
- \( k'_1 = (0.5, -0.8) \) and \( k''_1 = (-0.5, -0.2) \), where \( k + k'_1 + k''_1 = 0 \).
- \( k'_2 = (0.5, -0.9) \) and \( k''_2 = (-0.5, -0.1) \), where \( k + k'_2 + k''_2 = 0 \).

Then the difference between the frequencies \( \omega_1 \) and \( \omega_2 \) of resonant interactions are
\[
\omega_1 - \omega_2 = \omega(k'_2) - \omega(k'_1) + \omega(k''_2) - \omega(k''_1) = \frac{\partial \omega}{\partial k_y}(k''_1)(0.1) + \frac{\partial \omega}{\partial k_y}(k'_1)(-0.1) > 0.
\]

(Right) In addition, high frequency part is nonlinearly excited by the first three-wave coupling while low frequency part is nonlinearly damped by the second one.
**Why nonlinear frequencies between density and potential separate? (qADI)**

- For the potential, the basic nonlinear interaction is the same as in ADI.
  
  \[
  \langle \phi N \rangle : (k_x, k_y) = (+000,+015) 
  \]

- As \( \alpha \) decreases, the density advection becomes more significant. The density advection cascades energy forward. This process takes through lower frequency channel.
  
  \[
  \langle \psi N \rangle : (k_x, k_y) = (+000,+015) 
  \]
Density advection cascades energy forward through low frequency channel and results in low nonlinear frequency as $k_y$ increases.

Electrostatic nonlinearity triggers “inverse cascade” which energy goes nonlinearly toward the lower $k$ via nonlocal interaction in the micro-turbulence context. The process prefers low frequency channel to high frequency channel and results in nonlinear frequency shift to higher frequency as $k_y$ increases.

As $\alpha$ decreases, the density advection becomes dominant nonlinear process, which leads to frequency around zero at large $k$. 
Instead of multiplying by a conjugate, simply rearrange the equation in the $\omega - k$ space,

$$\Phi(\omega, k) = -\frac{1}{i\omega + L_k} \int N(k, k') \Phi(\omega', k') \Phi(\omega - \omega', k - k') dk' d\omega' = R(\omega, k) S(\omega, k).$$

This spectral function $\Phi$ can be a linear response function $R = -(i\omega + L_k)^{-1}$ with a source function

$$S(\omega, k) = \int N(k, k') \Phi(\omega', k') \Phi(\omega - \omega', k - k') dk' d\omega'.$$

or

$$P(\omega, k) = |\Phi(\omega, k)|^2 = |R(\omega, k)|^2 |S(\omega, k)|^2 = \Re R(\omega, k) \hat{N}(\omega, k)$$

With monochromatic source $S(\omega, k) = S_0(k) \delta(\omega - \omega_0)$,

$$P(k) = \int P(\omega, k) d\omega = \frac{|S_0(k)|^2}{i\omega_0 + L_k}.$$

With random-phase white noise $S(\omega, k) = S_0(k) \exp(-i\theta_w)$,

$$P(k) = \int P(\omega, k) d\omega = \int \frac{|S_0(k)|^2}{i\omega + L_k} d\omega.$$
In a multi-field equation,
\[
\frac{\partial \phi}{\partial t} = \bar{L}\phi + \bar{N}
\]  
then, in the Fourier transform,
\[
-i\omega\phi = \bar{L}\phi + \bar{N}
\]

Since $\bar{L}$ is non-Hermitian matrix, the orthogonality satisfies $\bar{U}\bar{L}\bar{V} = \bar{\lambda}$ where $\bar{U}$ and $\bar{V}$ are left and right eigen-matrices with the eigenvalue matrix $\bar{\lambda}$. The linear matrix $\bar{L}$ can expressed using the orthogonality relation, $\bar{U}\bar{V} = \bar{I}$
\[
\bar{L} = \bar{U}^{-1}\bar{\lambda}\bar{V}^{-1} = \bar{V}\bar{\lambda}\bar{U}.
\]

the HW equations are
\[
\phi = \bar{V}\left(-i\bar{\omega} - \bar{\lambda}\right)^{-1}\bar{U}\bar{N}
\]
where $\bar{\omega} = \omega\bar{I}$. 
for the HW model, without loss of generality, we can define $\tilde{V}$ and that, in turn, determines $\tilde{U}$ as an inverse of $\tilde{V}$,

$$
\tilde{V} = \begin{pmatrix} 1 & 1 \\ \beta_1 & \beta_2 \end{pmatrix} \quad \text{and} \quad \tilde{U} = \frac{1}{\beta_2 - \beta_1} \begin{pmatrix} \beta_2 & -1 \\ -\beta_1 & 1 \end{pmatrix}
$$

(6)

where $\beta_i$ represents $\psi/\phi$ for an eigenvalue $\lambda_i = -i\omega_i$.

$$
\left( -i\bar{\omega} - \bar{\lambda} \right)^{-1} = \begin{pmatrix} -i(\omega - \omega_1) & 0 \\ 0 & -i(\omega - \omega_2) \end{pmatrix}^{-1} = i \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\omega - \omega_2} \end{pmatrix}.
$$

(7)

Then

$$
\phi = \frac{i}{\beta_2 - \beta_1} \left[ \frac{1}{\omega - \omega_1} (\beta_2 N_\phi - N_\psi) + \frac{1}{\omega - \omega_2} (-\beta_1 N_\phi + N_\psi) \right]
$$

$$
\psi = \frac{i}{\beta_2 - \beta_1} \left[ \frac{\beta_1}{\omega - \omega_1} (\beta_2 N_\phi - N_\psi) + \frac{\beta_2}{\omega - \omega_2} (-\beta_1 N_\phi + N_\psi) \right]
$$

or

$$
\phi = \frac{i}{\beta_2 - \beta_1} \left[ \left( \frac{\beta_2}{\omega - \omega_1} + \frac{-\beta_1}{\omega - \omega_2} \right) N_\phi + \left( \frac{-1}{\omega - \omega_1} + \frac{1}{\omega - \omega_2} \right) N_\psi \right]
$$

$$
\psi = \frac{i}{\beta_2 - \beta_1} \left[ \left( \frac{\beta_1 \beta_2}{\omega - \omega_1} + \frac{-\beta_1 \beta_2}{\omega - \omega_2} \right) N_\phi + \left( \frac{-\beta_1}{\omega - \omega_1} + \frac{\beta_2}{\omega - \omega_2} \right) N_\psi \right]
$$
Role of linear modes (or responses)

- In the frequency space, it should be noted there is no more linear (unstable or stable) modes. It only represent a linear response. So the terms \((\omega - \omega_i)^{-1}\) are a “linear response”.

- We take ADI, so \(|N_\phi| \gg |N_\psi|\). Only the consideration of \((\omega - \omega_1)\) leads to

  \[ \frac{\psi}{\phi} = \beta_1 \]

  which implies that over all frequency range, the phase difference in the steady state is the one of the linearly unstable modes.

- In ADI, however, \(|\beta_2| \gg |\beta_1|, |\gamma_2| \gg |\gamma_1|\). Near \(\Re \omega_1\), \(\delta \omega = \omega - \Re \omega_1\)

  \[ \frac{\psi}{\phi} = \beta_1 \left( 1 - \frac{\omega - \omega_1}{\omega - \omega_2} \right) \simeq \beta_1 \left( 1 - \frac{\delta \omega - i\gamma_1}{(\Re \omega_1 - \Re \omega_2) - i\gamma_2} \right) \simeq \beta_1 e^{\frac{-\delta \omega - i\gamma_1}{(\Re \omega_1 - \Re \omega_1) - i\gamma_2}} \]

  \[ \arg(\phi \psi^*) \simeq -\arg(\beta_1) + \frac{\gamma_2 \delta \omega}{(\Re \omega_1 - \Re \omega_2)^2 + \gamma_2^2} \]

  therefore, \(\gamma_2 < 0\) confirms the observation, \(\arg(\phi \psi^*) \delta \omega < 0\), in the simulations.

- When density advection \(N_\psi\) becomes significant, this relation is expected to be violated.

  Let’s check this with the simulation data.
Even though the phase of large $k$ does not match the linear estimate at the linear eigen-frequency, the phase differences at the nonlinear frequency (or the most energetic frequency for each $k$) matches the linear relation from the previous slides.
In qADI, the phase relation in the simulation matches the linear relation, however not as well as in ADI.

Only approximately half $\omega > \tilde{\omega}$ follows the linear relation. Why? because the lower side of frequency spectrum is nonlinearly influenced by the density advection.
Discussion and Future direction

Discussion

● There are distinct asymmetry in the frequency spectrum of each $k$ and the linear and nonlinear interactions. In ADI and qADI, this asymmetry results smaller density flux in the high frequency side and larger density flux in the low frequency side.

● In ADI, nonlinear frequency shift and the cross-phase leads to the energy damping in the linearly unstable wavenumber. That is, nonlinear frequency shift can provide a way to damp energy without frequency broadening.

● At a energy-dominant wavenumber where $\omega \sim \omega_\ell$, the phase relation is not necessarily described by the one of the linearly unstable modes. **Stable mode** (or stable part of linear response, not rigorously) needs to complete the picture of density vs. potential fluctuation, therefore turbulent density flux.

● In the wave(weak) turbulence, the nonlinear frequency needs to be clearly incorporated properly into any closure or a subgrid model.

Future Direction

● The **distance between the linear frequency weighted by its phase and growth rates** could be a good measure for a broad frequency spectrum.

● Nonlinear response function should be more rigorously developed with use of two-point closure so that the phase can be predicted between $N_\phi$ and $N_\psi$.

● More frequency analysis should be studied in more general fluid model and kinetic model.