Subcritical Onset of Plasma Fluctuations and Magnetic Self-Organization in a Line-Tied Screw Pinch

by

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Abstract

The line-tied screw pinch is an important model for solar plasma and has been studied theoretically and numerically for decades. Often these theoretical models used current profiles and equilibria that are difficult to make using inductive techniques conventionally used for creating pinch plasmas.

This dissertation investigates the MHD stability of the line-tied screw pinch for a range of novel current profiles never before studied in a laboratory plasma. These studies used the Line-Tied Reconnection Experiment, a versatile line-tied screw pinch. The device was heavily modified to create more astrophysically relevant plasmas and provide increased diagnostic access. Multi-dimensional arrays of magnetic probes were built allow simultaneous measurements of dynamic plasma structures. Plasma is injected into the experiment at six discrete locations.

Stability analysis of the line-tied screw pinch assumes one dimensional equilibria. Internal measurements suggest that the 1D assumption is approximately valid for experimental plasmas, but plasmas undergo complex, three dimensional dynamics and self-organization without the presence of linear instability. Screw pinch equilibria with zero-net-current and hollow current profiles are created for the first time in the laboratory. The zero-net-current equilibrium transitions to a sub-critical dynamic state that reorganizes the equilibrium and prevents the formation of linear instability. The hollow current equilibrium self-organizes into a relaxed state at a critical value of magnetic field, independent of plasma current. Plasmas with two and three distinct flux ropes do not have a 1D equilibrium and exhibit dynamic interactions. These interactions create an inverse cascade as flux ropes become unstable and merge.
If we knew exactly the laws of nature and the situation of the universe at the initial moment, we could predict exactly the situation of that same universe at a succeeding moment. But, even if it were the case that the natural laws had no longer any secret for us, we could still only know the initial situation approximately. If that enabled us to predict the succeeding situation with the same approximation, that is all we require, and we should say that the phenomenon had been predicted, that it is governed by laws. But it is not always so; it may happen that small differences in the initial conditions produce very great ones in the final phenomena. A small error in the former will produce an enormous error in the latter. Prediction becomes impossible, and we have the fortuitous phenomenon.

Henri Poincaré [1]
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Ellen Zweibel, Jan Egedal, John Sarff, Carl Sovinec, and Chris Hegna were always willing to discuss data, ideas, and technical problems with my work. Thank you for all your advice and guidance as I worked to understand the complicated problems associated with performing and understanding plasma physics experiments and simulations.

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Finally, thank you Jennifer. It’s been a whirlwind adventure; you’ve always been by my side. I look forward to many more adventures with you.
**Definitions**

MHD  Magnetohydrodynamics

the device  The Line-tied Reconnection Experiment at the University of Wisconsin-Madison

$q(r)$  The safety factor experimentally measured at radius $r$

$q(a)$  The safety factor experimentally measured at the edge of the screw pinch

$q_{\text{crit}}$  The critical safety factor for instability onset

$I_p$  The total plasma current driven by electrostatic bias between electrodes

$L$  The device length

$m$  The (integer) azimuthal wavenumber of an MHD mode

$k_z$  The axial wavenumber of an MHD mode

$\gamma$  The linear growth rate of a mode

$B_z$  The axisymmetric guide field applied by external solenoids

$B_0$  The axisymmetric equilibrium field ($B_z$ and $B_\theta$)

$\delta B_z$  The $m = 0$ global perturbation to $B_z$ due to plasma diamagnetism

$\tilde{B}_z$  The $m = 0$ localized perturbation to $B_z$ by localized coils

G  Gauss, a unit of magnetic inductance equal to $10^{-4}$ Tesla

A  Ampere, a unit of Electrical Current
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Chapter 1

Introduction

What determines the stability of the line-tied screw pinch? If traditional theories of pinch stability require one dimensional equilibria, do experiments match those equilibria? What are the results of small deviations from laminar, 1D equilibria? How do dynamic line-tied plasmas relax and self-organize?

The history of magnetic pinches dates to 1905, when Pollock and Barraclough [2] investigated the collapse of a copper tube after it had been struck by lightning. They surmised that the large electrical current interacting with its own magnetic field crushed the lightning rod due to the $\mathbf{J} \times \mathbf{B}$ Lorentz force. Later, Bennett [3] applied a similar analysis to an electron beam in a diffuse gas, showing that the interaction between the electron current and its magnetic field could counteract radial pressure, compressing and focusing the electron beam.

Bennett’s discovery illuminated what appeared to be a straightforward path toward fusion energy. The high temperatures and pressures required for fusion would be realized in a plasma confined with magnetic pinch forces. A wave of linear experiments in the 1940’s and 1950’s intended to prove the magnetic pinch fusion concept. These experiments failed to produce sustainable reactions. Fusion research moved way from linear devices and toward to toroidal devices with more complicated magnetic geometries.

The magnetic pinch, in both linear and toroidal geometries, has been the focus of extensive fusion research in the decades since these early efforts. The concept has been applied to a number
of astrophysical phenomenon and can be used to study many basic plasma physics topics. The general purpose of this dissertation is to present results from a linear screw pinch experiment with application to fusion research, astrophysics, and basic plasma phenomena.

The organization of this Chapter follows: Second 1.1 introduces the theory of magnetohydrodynamics. Section 1.2 discusses the equilibrium, stability, and history of the magnetic screw pinch. Section 1.3 introduces magnetic reconnection, while Section 1.4 discusses the mathematical ideas of chaos, turbulence, and self-organization. Section 1.5 discusses outstanding questions in current fusion research and their connection to the linear screw pinch. Next, Section 1.6 presents several models for astrophysical phenomenon that utilize the screw pinch geometry. Section 1.7 discusses current research in the field and Section 1.8 discusses past results from the Line-Tied Reconnection Experiment. Finally, Section 1.9 presents goals for the dissertation and an outline of this document.

1.1 Magnetohydrodynamics

The majority of this dissertation will consider the behavior of plasmas within the framework of magnetohydrodynamics (MHD). MHD is a single-fluid approximation to plasma behavior. It assumes that the plasma is highly collisional with Maxwellian particle distributions; that the scales of interest are much larger than scales such as the ion skin depth, the Larmor radius; and that plasma evolution is much slower than the ion cyclotron time. Using these approximations, the kinetic description of plasmas can be reduced to the following set of equations, representing Resistive MHD:
\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \]  \hspace{1cm} (1.1)

\[ \rho \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \right) \mathbf{v} = \mathbf{J} \times \mathbf{B} - \nabla p \]  \hspace{1cm} (1.2)

\[ \frac{d}{dt} \left( \frac{p}{\rho \gamma} \right) = 0 \]  \hspace{1cm} (1.3)

\[ \mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{J} \]  \hspace{1cm} (1.4)

\[ - \frac{\partial \mathbf{B}}{\partial t} = \nabla \times \mathbf{E} \]  \hspace{1cm} (1.5)

\[ \mu_0 \mathbf{J} = \nabla \times \mathbf{B} \]  \hspace{1cm} (1.6)

\[ \nabla \cdot \mathbf{B} = 0 \]  \hspace{1cm} (1.7)

The first three equations represent mass continuity, force balance, and energy conservation, the fourth equation is a generalized Ohm’s law, and the last three equations are the low frequency approximation to Maxwell’s equations. \( \rho \) is the mass density of the plasma, \( \mathbf{v} \) is fluid velocity, \( p \) is fluid pressure, \( \mathbf{J} \) is current, \( \mathbf{B} \) is magnetic field, \( \mathbf{E} \) is electric field, and \( \eta \) is electrical resistivity.

In most plasmas, \( \eta \) is very low and be approximated a zero. In that case, the above equations reduce to Ideal MHD, which is very useful for studying the equilibrium and stability of macroscopic plasmas. Combining Eqns. 1.4, 1.5, and 1.6 we can derive the magnetic induction equation

\[ \frac{\partial \mathbf{B}}{\partial t} = \nabla \times \mathbf{v} \times \mathbf{B} + \frac{\eta}{\mu_0} \nabla^2 \mathbf{B} \]  \hspace{1cm} (1.8)

If \( \eta \) is small, we can ignore the final term. The resulting equation implies that magnetic field lines are “frozen-in” to the moving plasma in ideal MHD. This means that magnetic forces can move the fluid can fluid motion can, in turn, move magnetic field. The most important implication of this equation is that fluid associated with one magnetic field line is topologically separated from fluid on another field line.
1.2 The Screw Pinch

The equilibrium of the magnetic pinch can be calculated using Ideal MHD. Assuming that the equilibrium doesn’t vary in z or $\theta$ and that flows are small, the force balance equation becomes:

$$\frac{\rho \partial v}{\partial t} + \rho \vec{v} \cdot \nabla \vec{v} = \vec{J} \times \vec{B} - \nabla p$$  \hspace{1cm} (1.9)

$$\frac{\partial}{\partial r} \left( p + \frac{B_{\theta}^2 + B_z^2}{2\mu_0} \right) + \frac{B_{\theta}^2}{\mu_0 r} = 0$$  \hspace{1cm} (1.10)

Thus, the pressure profile is set by the axial and azimuthal magnetic field. One can consider three types of pinch equilibria.

The first, called the theta pinch (Figure 1.1), is formed when external solenoids and azimuthal plasma current create a purely axial magnetic field with no azimuthal magnetic field, i.e.:

$$\frac{\partial}{\partial r} \left( p + \frac{B_z^2}{2\mu_0} \right) = 0$$  \hspace{1cm} (1.11)

The theta pinch compresses the plasma uniformly in $z$, creating a stable equilibrium. Early exper-
ments on the theta pinch revealed excellent plasma performance. Rapid losses on either end of
the cylinder limited the lifetime of the plasma, preventing use as a fusion device. Unfortunately,
the theta pinch cannot be wrapped into a toroidal geometry - the stronger magnetic field at the
center of the torus pushes the plasma outward and disrupts the equilibrium. Despite excellent early
performance, the theta pinch was abandoned as research moved to toroidal experiments.

The second geometry, called the z pinch (Figure 1.2), is formed when axial plasma current
creates an azimuthal magnetic field with no axial field, i.e.:

\[
\frac{\partial}{\partial r} \left( p + \frac{B_\theta^2}{2\mu_0} \right) + \frac{B_\theta^2}{\mu_0 r} = 0
\]  

(1.12)

The z pinch can be wrapped into a torus without loss of equilibrium; linear and toroidal z pinch
experiments were common in the early days of fusion research. However, the z pinch is very
unstable to the m=0 sausage mode. In the sausage mode, a perturbation compresses the plasma
at one z location. Due to the compression, current density and magnetic field rise, increasing the
pinch force. If plasma pressure cannot support the increased pinch force, the perturbation will
grow exponentially, rapidly choking the current path and disrupting the plasma.
The toroidal incompatibility of the theta pinch and the instability of the z pinch led to the development of a mixed concept called the screw pinch (Figure 1.3). The screw pinch involves both axial and azimuthal currents and magnetic fields, i.e.

$$\frac{\partial}{\partial r} \left( p + \frac{B_\theta^2 + B_z^2}{2\mu_0} \right) + \frac{B_\theta^2}{\mu_0 r} = 0$$

(1.13)

This means magnetic field lines twist (or screw) around the plasma. The axial magnetic field stabilizes the sausage mode while the azimuthal magnetic field allows the screw pinch to be wrapped into a torus. Two prominent fusion concepts, the tokamak and the reversed field pinch, are toroidal screw pinches with different $B_\theta/B_z$ ratios. When the toroidal major radius $R_0$ is much larger than the minor radius $a$, the cylindrical screw pinch is an excellent model for the equilibrium and stability of toroidal fusion devices.

### 1.2.1 Screw Pinch Stability

While the screw pinch is generally stable to the sausage mode, it can be unstable to another Ideal MHD mode: the kink. The kink is an $m=1$ mode where the plasma deflects from its cylindrical
Figure 1.4: Illustration of Kink Mode. (a) shows the destabilizing factor of the azimuthal field, (b) shows the stabilizing factor of the axial field.

There are two types of kink modes. The first, called the external kink mode, happens when the entire plasma column shifts. When this shift is large enough, the plasma will hit the wall of the device, disrupting plasma operation. The second kink, called the internal kink, occurs when some surface inside the plasma shifts, but the edge of the plasma stays stationary.

The physical mechanism of the kink is illustrated in Figure 1.4. Assume a small deflection from the cylindrical equilibrium. This deflection brings azimuthal field lines closer together on one side of the plasma, while pushing them apart on the other side, as shown in Figure 1.4 (a). This increases the magnetic pressure below the plasma and decreases magnetic pressure above the plasma, pushing the equilibrium further out. Fighting this force is the axial magnetic field, which fights to restore the cylindrical equilibrium via magnetic tension, as shown in Figure 1.4 (b). The competition between these two forces determines the stability threshold of the kink mode.

Kruskal and Tuck [5] and Shafranov [6] independently derived the stability threshold for the external kink mode for a cylindrical screw pinch equilibrium with uniform current density and periodic boundary conditions. After linearizing the ideal MHD equations, they found a mode with
form
\[ \tilde{f}(r, \theta, \phi, t) = f(r)e^{im\theta + in\phi + \gamma} \] (1.14)

where \( \tilde{f} \) represents some fluctuating quantity. This mode is unstable when the safety factor drops sufficiently low:

\[ q(a) = \frac{aB_z}{RB_\theta} < \frac{m}{n} \] (1.15)

where \( a \) is the radius of the plasma and \( R \) is the length of the cylinder divided by \( 2\pi \), i.e. the major radius of a torus. The most unstable mode is the \( m = 1, n = 1 \) external kink mode, unstable at

\[ q(a) < 1, \]  

with growth rate proportional to the crossing time of an Alfvén wave. This fast growth rate means that the external kink disrupts plasma operation faster than can be mitigated.

If \( q(a) > 1 \) but \( q(0) < 1 \), the periodic screw pinch is unstable to the Ideal MHD internal kink mode [7]. The internal kink mode is an \( m = 1 \) instability that moves the plasma located at \( r < r_0 \) where \( r_0 \) is the location of the resonant surface where \( q(r_0) = 1 \). The non-linear evolution of this mode create a singular gradient in \( B \) and a strong current sheet at the resonant surface [8]. This sharp current drives magnetic reconnection (described in Section 1.3), reorganizing the plasma equilibrium and causing the classic tokamak sawtooth [9, 10]. This process is illustrated in Figure 1.5.

The analysis of the linear screw pinch with finite length is more complicated. First, the finite-length safety factor is defined:

\[ q(r) = \frac{2\pi rB_z}{LB_\theta} \] (1.16)

Stability depends on the axial boundary conditions. For the purposes of this dissertation we will consider two magnetic boundary conditions. The first, called line-tying, occurs when magnetic field lines are fixed to the boundary. In astrophysical situations, this occurs at sharp density transitions between magnetically dominated plasma and dense, pressure dominated plasma. In laboratory systems line-tying is enforced by highly conductive electrodes at the end of the cylinder. In the second boundary condition, the magnetic field can be free to slip at one end of the plasma. Sheath
resistivity and other non-ideal effects can cause this boundary condition in laboratory systems.

In contrast to the periodic case, the stability of the finite-length screw pinch cannot be described by a single mode. The boundary conditions require the interaction between several modes, i.e.:

$$\tilde{f}(r, \theta, z, t) = \sum_j (f_j(r)e^{im_j + ik_jz + \gamma_j t})$$ (1.17)

The stability of line-tied screw pinch is not analytically solvable for a general case. If one assumes uniform current density and a large aspect ratio ($L \gg a$), the boundary conditions can be satisfied with an $m = 1$ perturbation of the form:

$$\tilde{f}(r, \theta, z, t) = f(r)e^{i(k_1 + k_2)z/2 + \gamma t} \sin \left(\frac{n\pi z}{L}\right)$$ (1.18)

This perturbation is a multi-wavelength external kink mode. As in the toroidal case, it is unstable when $q(a) < 1$ [12, 13]. Similarly, if one end of the plasma is line-tied but the other end freely
slips, the external kink mode is unstable when \( q(a) < 2 \) [14].

For more general equilibria with lower aspect ratios, understanding the stability of the kink mode requires numerical analysis [15, 16]. The kink mode is destabilized by moderate values of electrical resistivity [17, 18], and plasma flow can be stabilizing or destabilizing depending on the profiles of the equilibrium [19]. The details of these effects are highly dependent on the specific equilibrium in question.

The internal kink mode with line-tied boundaries is similarly complicated. Huang et al. [20] studied the stability of a particular equilibrium with a semi-analytical code. Importantly, they saw the line-tied internal kink drive a strong current sheet despite the fact that line-tied systems don’t have the resonant surface associated with current sheets in periodic systems. Many studies of the line-tied internal kink mode have been performed for solar-relevant equilibria. These results will be discussed in detail in Section 1.6 and Chapter 4.

1.3 Magnetic Reconnection

There are some situations where even very small \( \eta \) can play a dynamic role in plasma evolution. Consider, for example, the situation where some external driving force or instability pushes two magnetic field lines together, as illustrated in Figure 1.6 (a). This compression of magnetic field increases \( \nabla^2 B \) in Equation 1.8, driving a strong, thin current sheet and increasing the effect of resistivity. Resistivity then breaks field lines, mixing the plasma and pushing it out of the current sheet.

Sweet [21] and Parker [22] developed a model for how this occurs using resistive MHD. Their analysis showed that the reconnection rate is proportional to \( S^{-1/2} \), where \( S = \sqrt{\mu_0/\rho} \frac{L B}{\eta} \). The Sweet-Parker model of reconnection allows relatively fast dissipation of magnetic energy and plasma mixing, but is too slow to explain observations of solar flares. More advanced models attempt to explain the speed of reconnection events in fusion devices and solar flares, but these models are beyond the scope of the experiments in this dissertation.
1.4 Turbulence, Self-Organization, and the Loss of MHD Equilibrium

Resistive MHD, as described above is a complicated set of nonlinear partial differential equations with dissipation. Resistive MHD can support laminar dynamics, such as the equilibria and linear instabilities presented in Section 1.2, and a number of waves and periodic systems.

Under other conditions, resistive MHD can transition to chaos. The classic example of fluid chaos is turbulence. Traditional fluid turbulence occurs when eddies are driven in large scale flow. Eddies then interact with each other, shearing and transferring energy to smaller scales. These smaller scales then repeat the process, transferring energy to ever smaller scales. Eventually, fluid viscosity dissipates the energy at the smallest scales. Turbulence is intermittent and broadband in both frequency and space.

Turbulent eddies can be created critically, when the large scale flow or magnetic structure in
a plasma becomes unstable. They can also be driven or sub-critically, when created external or
boundary-driven fluctuations. Sub-critically driven turbulence can even modify the equilibrium
profiles, preventing the development of linear instabilities [23].

Turbulence, like other chaotic systems, can self-organize. Turbulent interactions can generate
large scale, stable structures from the interactions of small scale fluctuations [23, 24, Ch. 15]. In
MHD turbulence, for example, systems driven at small scales with a large background magnetic
field tend to self-organize into large-scale, three dimensional structures [25]. This processes is
called the “inverse cascade.” The inverse cascade transfers energy from small scales to large scales,
opposite what is expected from fluid turbulence.

Finally, the nonlinear evolution of instabilities and chaotic states such as turbulence lead to
the loss of MHD equilibrium where $J \times B \neq \nabla p$. This loss of MHD equilibrium has been clearly
demonstrated for the rotating nonlinear evolution of the external kink mode [26] and in the sudden
transition of slowly evolving solar plasmas [27]. Plasmas without MHD equilibrium are inherently
dynamic - they either rotate with coherent modes or exhibit turbulent fluctuations.

### 1.4.1 Turbulence in Dissipative Systems

In traditional fluid turbulence energy is injected at some large scale and removed by dissipation at
some smaller scale. These two scales are separately by an inertial range with well defined statistical
properties.

The experiments in this dissertation are dissipative than traditional turbulence. This means that
the separation between driving scales and dissipative scales may not be large enough to develop a
well defined inertial range. In this case, the traditional defintion of turbulence is not strictly valid.

Dissipative systems can, however, exhibit turbulence-like behavior. This can be mathemati-
cally defined as the chaotic interaction between fluctuations in the plasma. Such fluctuations are
qualitatively similar to turbulence but have different statistical properties. In general, the idea of
“turbulence” in this dissertation refers to this broader idea of chaotic interacting fluctuations, not
the well defined inertial range present in traditional turbulence.
1.4.2 Measurements of Chaos

Recent advances in the statistics of chaos allow for more subtle measurements of the complexity of time-series data [28, 29]. These techniques allow statistical distinction between, smooth deterministic systems (such as polynomial and sinusoidal functions), noise, and chaotic systems such as turbulence. These statistical techniques can help illuminate underlying physical processes by their effect on system behavior. These techniques have recently been applied to plasma measurements [30, 31, 32].

Here we uses the complexity–entropy plane, developed by Rosso et al. [29] to map the underlying processes in flux rope interactions. This method calculates two statistical measures of data and plots them relative to one another. The first measure, the Bandt-Pompe permutation entropy [28] measures the amount of randomness in a time signal. A time signal \( t \), \( \text{len}(t) = T \) is divided into segments \( x_i = \{ t_i, ..., t_{i+n-1} \} \), \( i = 0...T - n + 1 \) where segment length \( n \) is a chosen parameter. The permutation of each segment is then calculated, where each element in the segment is labeled in ascending order. There are \( N = n! \) such permutations. For example, the array \( \{ 5, 1, 2, 3, 1, 4, 7, 0 \} \) has permutation \( \pi = \{ 5, 2, 3, 4, 1 \} \). The probability of each permutation is then calculated:

\[
p(\pi_j) = \frac{\#x_i | x_i \text{ has permutation } \pi_j}{T - n + 1}
\] (1.19)

The permutation entropy is then:

\[
S(p) = -\sum_{j}^{N} p(\pi_j) \ln(p(\pi_j))
\] (1.20)

which measures the randomness of patterns in a signal.

Permutation Entropy alone cannot distinguish between stochasticity and chaos. For this, we must define the Jensen-Shannon Complexity [29]:

\[
C_{js} = -2 \frac { S \left( \frac{p+p_e}{2} \right) - \frac{1}{2} S(p) - \frac{1}{2} S(p_e) } { \frac{N+1}{N} \ln(N+1) - 2 \ln(2N) + \ln(N) } \frac{S(p)}{\ln(N)}
\] (1.21)
Figure 1.7: The Complexity-Entropy Plane for \( n = 5 \) with several signal maps plotted. The solid lines indicate minimum and maximum complexity for a given entropy. Three signals are plotted: the square is a Sine Wave, the upwards triangle is the chaotic Henon map, and the downwards triangle is the Chaotic Logistic Map. Gaussian noise falls at \( C=0, S=1 \). The dashed line corresponds to Fractional Brownian Motion and serves as an approximate separator between chaos at high complexity and stochasticity at low complexity.

where \( p_e \) is the state where all permutations have equal probability. The Jensen-Shannon Complexity measures larger correlations in the signal. Together, these two measurements provide a way to distinguish between types of processes. Smooth processes, such as polynomials and sinusoidal functions, have low entropy but high complexity. Stochastic processes, such as noise, have high entropy but low complexity. Finally, chaotic process have medium-high entropy and high complexity. Thus, mapping signals onto this plane allows us to distinguish between various physical processes.

The location of various deterministic, stochastic, and chaotic processes in the C-S plane are shown in Figure 1.7. Deterministic processes, such as sine waves, fall in the bottom left corner of the graph, while chaotic processes fall at high complexity and medium entropy. White noise (a completely stochastic process) falls at the bottom right of the plot. Fractional Brownian Motion (a correlated random walk process) marks a rough boundary between chaos and noise.
Figure 1.8: The Current Hole Equilibrium measured on JT-60U for a number of times in the discharge. Note that $j = 0$ for $\rho < 0.2 - 0.4$, depending on time, and the safety factor rises to infinity on axis. Figure from Fujita et al. [33].

### 1.5 Experimental Connections to Fusion

Researchers have spent decades modeling high-aspect ratio fusion devices with the screw pinch. Because of this theoretical work, easy to perform screw pinch experiments can be directly compared to fusion models. Two topics relevant to the fusion community are considered.

#### 1.5.1 Resistive Wall Mode

The Resistive Wall Mode is an external kink-like mode that limits the high $\beta$ operation of fusion devices. It occurs when magnetic flux from the mode slowly penetrates the resistive shell of the experiment, allowing the kink to grow. If a second resistive shell moves at high speed just outside the first, the two wall system acts as a perfect conductor, stabilizing the resistive wall mode [34, 12]. Paz-Soldan [35] studied the stabilization of the Resistive Wall Mode by a rotating, conducting
second wall. His results are briefly summarized in Section 1.8.2.

1.5.2 Current Holes

In early 2000’s, a number of fusion devices began experimenting with off-axis current drive by neutral beam, radio-frequency waves, and the bootstrap current. These off axis current drives induce a negative toroidal electric field at the core, creating a region with no current at the center of the experiment. These “current hole” equilibria have been observed on JT-60U [33], JET [36, 37], ASDEX-U [38], and DIII-D [39]. An example of this equilibrium is shown in Figure 1.8.

In certain cases, off axis current drive was so strong that on-axis current should have been driven negative. All experiments performed to date show that the current “clamps” to zero, never becoming negative. This is interpreted as a self-organized system, where self-driven flows or the repetitive action of the resistive kink mode clamp the current on axis to zero [40].

Interestingly, the line-tied hollow current equilibrium also exhibits self-organized behaviors. The line-tied hollow current equilibrium is studied in Chapter 5.

1.6 Experimental Connections to Astrophysics

The screw pinch shares many similarities with the astrophysical concept of a flux rope. A flux rope is a structure of plasma and twisted magnetic field. Depending on geometry, flux ropes can be cylindrical, toroidal, or have complicated three-dimensional magnetic field lines that may or may not close. In their most basic form, flux ropes are finite-length screw pinch equilibria that can become unstable and interact with other magnetic structures. Flux ropes are found in the solar corona, the solar wind, and the earth’s magnetosphere [41].

1.6.1 Coronal Loops and Solar Flares

Flux rope structures in the solar atmosphere are called coronal loops. The corona is a hot \((T_e > 10^6 \text{ K})\), diffuse \((10^{15-16} \text{ m}^{-3})\) plasma surrounding the sun. Coronal loops contain magnetic fields (100
which dominate the movement of plasma. At the base of the corona there is a transition region where plasma density increases by a factor $10^{7-8}$ leading to the pressure dominated photosphere. This transition is less than 1% of the height of the corona. This sudden transition from the magnetically dominated corona to the pressure dominate photosphere “ties” coronal loops to the solar surface.

Coronal loops can flare, releasing up to $10^{25}$ Joules of energy in tens of minutes [42]. Coronal loops tend evolve slowly over the course of days or weeks and then impulsively flare. The cause of this sudden flaring is still unknown, but these turbulent reconnection events heat the corona, similar to the Parker model of nanoflares [43, 44] magnetohydrodynamic instabilities may be responsible. Observations cannot make the detailed magnetic measurements required to calculate the stability of coronal loops. Several models have been suggested for solar flares, but few have been tested.

1.6.2 Zero-Net-Current Coronal Loops

Perhaps the simplest model in which magnetic energy is store in coronal loops is the twisting of magnetic fields by footpoint motion. Gold and Hoyle [45] suggested that magnetic field lines in a coronal loop may be twisted by turbulent photospheric motions at the “line-tied” ends of the loop. Since twisting motions are slower than the Alfvén time, the system evolves in a quasi-static equilibrium. This drives current along the loop, increasing free magnetic energy. At some point the loop becomes unstable, driving magnetic reconnection and releasing the stored magnetic energy as a solar flare.

The details of this process depend critically on the twisting profile at the foot points of the coronal loop. Interestingly, if the twisting motion is smaller than the magnetic field, the coronal loop has “zero-net-current.” This concept is illustrated in Figure 1.9 and Figure 1.10.

The zero-net-current equilibrium is unconditionally stable to the external kink. There is no free energy for mode growth in the current free magnetic field at the edge of the plasma. The internal kink, however, can be unstable. The stability criterion cannot be analytically determined,
Figure 1.9: Magnetic Field lines in a zero net current coronal loop. The field lines are locally straight at the edge of the plasma, indicated that there is no net current in the loop.

Figure 1.10: An example photospheric vortex velocity profile and the resulting current profile. The localized twist creates a coaxial current structure with positive current on axis and negative current around it. $r$ is the minor radius of the coronal loop.
so researchers turned to numerical simulations. Coronal loops have high aspect ratio ($L/a \approx 10$) so most early simulations used cylindrical geometry. Many authors have studied these equilibria, finding the presence of the internal kink instability and non-linearly driven magnetic reconnection [46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 20]. Other authors showed that this equilibrium can be turbulent, changing the stability and dissipation properties of the loop [56, 57, 58, 23]. The zero net current equilibrium will be further studied in Chapter 4.

1.6.3 Interaction of Flux Ropes

Many magnetic structures in the corona and other astrophysical systems can be viewed as collections of many interacting flux ropes. Parallel flux ropes attract each other through $\mathbf{J} \times \mathbf{B}$ forces and repel each other through magnetic pressure and tension [59, 60]. The interaction of these forces creates complex dynamic systems, including magnetic reconnection [61, 62, 63], the merger of flux ropes [64, 65], and bouncing of flux ropes [66]. Recent studies have shown that flux ropes exhibit chaos when they interact [31]. This has led to the increased use of statistical diagnostics when interpreting flux rope data and it’s connection to turbulence and chaotic systems. The interaction of multiple flux ropes is studied in Chapter 6.

1.6.4 The Coronal Heating Problem

Another outstanding problem in solar physics is the coronal heating problem. As discussed above, the corona is much hotter than the surface of the sun. This indicates it must be continuously heated by some mechanism. One possible mechanism for this heating is turbulence. Magnetic turbulence is driven in by instabilities and photospheric motions. This turbulence creates small scale magnetic reconnection events, continuously converting magnetic energy into heat. This model for heating the corona is called the nanoflare model [43, 44].
1.7 Connections to Current Research

A surge of astrophysical interest in the line-tied screw pinch in the last two decades has inspired a number of experimental investigation into the properties of line-tied screw pinches and flux ropes. Three devices are currently studying the linear screw pinch. These experiments primarily study the external kink mode and interactions between multiple screw pinches/flux ropes.

The Reconnection Scaling Experiment [67], or RSX, is a device at Los Alamos National Laboratory. RSX is capable of operating with one or two independent screw pinch equilibria and measures plasma parameters with a set of scannable probes. Using a single screw pinch/flux rope, the RSX team have thoroughly investigated the effects of flow and axial boundary conditions on the stability of the external kink mode [68, 69, 70, 71]. In particular, RSX confirmed the stability of the partially line-tied kink, as discussed in Section 1.2. More recent work focused on understanding the three dimensional force balance of the saturated line-tied kink mode [26]. Detailed three dimensional measurements of magnetic structure, plasma density, and temperature confirmed that the rotating, helical state of a kinking screw pinch does not obey traditional force balance (\(J \times B \neq \nabla p\)). This may be balanced by unmeasured inertial forces or it may be a dynamic system with no define equilibrium. Using two flux ropes, the RSX team have reported a variety of complex behavior. Two flux ropes were observed to collide and merge via magnetic reconnection [61] or collide and bounce in a fully three dimensional manner [66], depending on plasma conditions.

The Large Plasma Device [72], or LAPD, is a device at the University of California - Los Angeles. The LAPD was designed to create a large, magnetized plasma for the study of Alfven waves, but quickly began the study of interacting flux ropes [73]. LAPD uses a set of precision controlled probes and a very high plasma repetition rate (1 Hz) to create large scale, three dimensional data sets of flux rope interactions. Like RSX, these data sets require certain assumptions about shot-to-shot reproducibility. Using this method, researchers at the LAPD identified three dimensional reconnection sites between two flux ropes [62], and complex dynamics including reconnection, merging, and bouncing between three flux ropes [63]. During these studies researchers noticed that certain plasma discharges didn’t match shot-to-shot reproducibility. Statistical analysis re-
revealed complex and mathematically chaotic behavior in the interaction between kink-unstable flux ropes [31].

The third experiment studying the properties of the linear line-tied screw pinch is the subject of this dissertation; the Line-tied Reconnection Experiment (LTRX) at the University of Wisconsin - Madison. The LTRX is the only line-tied experiment studying the internal kink mode. LTRX can produce much higher current density and magnetic fields than LAPD and RSX. This allows it to study the strongly nonlinear behavior of the line-tied screw pinch. It also utilizes dense, multi-channel diagnostics avoiding the need for shot to shot averaging described above. The device is described in Chapter 2 while previous results from the device are discussed in Section 1.8.

1.8 Past Results On the Experiment

The experiments presented in this dissertation build on the work of three other graduate students: W. F. Bergerson [74], D.A. Hannum [75], and C. A. Paz-Soldan [35]. The results from these previous incarnations of the experiment are presented as a foundation for the work presented in this dissertation. Most of these results are more relevant to fusion experiments than astrophysical system but certain aspects inspired further astrophysically-motivated research.

D. A. Hannum and W. F. Bergerson completed most of the experimental construction. Starting with an empty room, Hannum and Bergerson built and installed vacuum systems, plasma creation and confinement schemes, power supplies, and a data acquisition system; many of these systems were modified or custom built during the course of device construction. Paz-Soldan et. al. [76] reported on state of the experiment near the beginning of this dissertation. Much of the hardware and control system was replaced as part of this dissertation. For updated experimental conditions, see Chapter 2.

1.8.1 Internal Kink Instability

Bergerson et al. [77] found high frequency coherent MHD modes in the device. As shown in Figure
Figure 1.11: (a) and (b) show the magnetic frequency spectrum of modes in plasmas carrying 5000 and 7000 Amps, respectively. (c) and (d) show the instantaneous magnetic eigenfunctions at two times separated by a half a period for the plasmas in (a) and (b), respectively. (c) shows a line-tied eigenfunction, while (d) shows a partially line-tied eigenfunction. Image from Bergerson et al. [77]

1.11, magnetic fluctuations were located at either the mid-plane or one end of the experiment, depending on plasma parameters. These modes were interpreted as line-tied internal kink modes. The modes became unstable when $q$ dropped below 1, as predicted by simulations of line-tied internal kinks. Other authors [78], however, have suggested that the mode in Figure 1.11 (d) may be an partially line-tied external kink mode unstable at safety factor $q = 2$ [14].

At very unstable plasma parameters, the current profile suddenly relaxes, as shown in Figure 1.12. These relaxations were interpreted as magnetic reconnection in the experiment, reminiscent of sawteeth in toroidal devices. This experiment utilized no internal diagnostics and direct observation of the reconnection geometry was impossible. Internal observation of reconnection events is one of the primary goals of this dissertation.

Later work by Hannum [75] made detailed maps of the temperature density of the plasma while Paz-Soldan et al. [79] attempted to quantify the saturated internal kink mode observed by Bergerson et al. [77]. They utilized a scanning magnetic probe, a stationary reference magnetic
Figure 1.12: (a) shows the plasma current profile just before and just after a reconnection event. (b) shows the corresponding rise in safety factor. Image from Bergerson et al. [77]

probe, and shot-to-shot reproducibility to build up 2.5 dimensional representations of the saturated kink mode. The plasma saturated into a three-dimensional helical equilibrium that rotated, stably, for the duration of the shot. In contrast to [77], Paz-Soldan et al. [79] observed no critical safety factor for the instability, shown in Figure 1.13. They postulated that pressure driven modes might play a role in the plasma.

### 1.8.2 The Resistive Wall Mode

Later work by Bergerson et al. [80] modified the radial boundary condition in the device by installing a variety of conducting vacuum vessels. Low frequency MHD instabilities occurred when edge $q < 1$ and grew at approximately the resistive diffusion time of the wall ($\tau_w$). This mode was interpreted as a Resistive Wall Mode.

Following theories on stabilizing the Resistive Wall Mode [34, 12], Paz-Soldan et al. [81] built and operated a rotating, conducting wall outside of the conductive vacuum vessel and studied its effects on the stability of the Resistive Wall Mode. The rotating wall stabilized the mode, as
Figure 1.13: (a)-(d) scaling of the frequency and amplitude of coherent fluctuations in the plasma. Instability occurs at a critical plasma current that varies only weakly with $B_z$, inconsistent with a critical safety factor. (e) and (f) show the characteristic frequencies in the device, and (e) shows that the mod is localized toward the anode. Image from Paz-Soldan et al. [79]
predicted by theory. The mode exhibited a critical safety factor $q_{\text{crit}} = 1.3$, slightly larger than the predicted $q_{\text{crit}} = 1$. Other work included extensive investigations of mode-locking of the instability [82] and the effect of error fields on mode stability [83].

At the beginning of this dissertation a borosilicate glass wall replaced the steel and copper walls used in the Resistive Wall Mode Studies. Thus, the rest of this dissertation will ignore the effects of radial boundary conditions on plasma stability.

1.9 Goals and Dissertation Outline

The Line-Tied Reconnection Experiment, discussed in Chapter 2 utilizes seven discrete current sources to make plasma. All previous experiments on the device relied on the assumption that these individual current channels merge and form smooth, one dimensional equilibria. How valid is this approximation? Internal measurements of current merger (Chapter 3) suggest that this approximation holds for most of the experimental volume, but small deviations from 1D equilibria are inherent in the system.

Do small deviations from 1D equilibrium result in small or large deviations from calculated stability? Asked another way, is the line-tied screw pinch chaotic? Chapters 4 and 5 seek to answer this question. Chapter 4 studies the zero-net-current equilibrium and compares data to models of coronal loop and solar flare evolution. Chapter 5 studies the stability of hollow current equilibrium and compares it to traditional kink theory.

Finally, how do fully three dimensional equilibria compare to approximately one dimensional equilibria? Chapter 6 studies the three-dimensional interactions of two and three flux ropes and compares their behavior to the measurements in the previous chapter.

Results are summarized and synthesized in Chapter 7. Unanswered questions and ideas for solving them are presented.
Chapter 2

The Line-Tied Reconnection Experiment

For over a decade a device has operated at the University of Wisconsin - Madison. This device was built to study the stability of the line-tied screw pinch with a variety of radial boundary conditions. The device, originally called the Rotating Wall Machine [76], was constructed by graduate students Will Bergerson and David Hannum and engineer Roch Kendrick. They characterized the plasma and studied the effect of insulating [77], resistive, and ferritic [80] boundary conditions on the stability of the plasma. Later, Carlos Paz-Soldan studied the internal structure of saturated instabilities [79] and the effect of differentially rotating boundaries on stability [81].

Earlier versions of the device have been described in detail by other graduate students [74, 75, 35]. At the beginning of this dissertation, the experiment was re-purposed to study more complex plasma equilibria with specific applications to astrophysical processes. Extensive device modifications were required to study these new topics. The device was renamed to the “Line-tied Reconnection Experiment” to reflect these hardware upgrades and the switch in experimental focus.

The device always suffered from poor confinement due to the open magnetic field-line geometry. This yields low electron temperatures and low Lundquist numbers making astrophysical studies difficult. To mitigate these effects, magnetic field was increased. To reach the same safety factors for instability, the experiment was lengthened from 1.22 m to 2.08 m. Four new solenoid
provide magnetic field for the new volume. The changes increased electron temperature moderately (20%) and decreased density significantly (75%). This yielded a 5-10x increase in the Lundquist number, making astrophysical comparisons slightly more relevant.

A detailed multidimensional probe array was designed and constructed as part of this dissertation. Custom analog integrators were designed and constructed to integrate and amplify the signals from the B probes. The probe array allows direct testing of equilibrium assumptions and detailed internal measurements of magnetic activity.

An outline of this chapter follows: Section 2.1 will describe the experimental The largest changes to the experiment made in this dissertation were a lengthening of the experimental cylinder hardware and computer control. Section 2.2 will describe the diagnostics used in this dissertation. Finally, Section 2.3 will discuss typical plasma parameters and the applicability of experimental results to astrophysical plasmas.

2.1 Machine Description

2.1.1 Vacuum Vessel

The primary experimental volume is a cylinder of length 2.08 m and diameter 19 cm. The volume is bounded by a set of borosilicate glass tubes with 5 mm wall thickness, shown in Figure 2.2, which serve as the vacuum vessel. Glass was chosen as the vessel material to minimize the influence of conducting boundaries on plasma stability and better mimic the boundary conditions occurring in astrophysical plasmas. Two large stainless steel “bells” are located at either of the cylinder. The bells hold the plasma source, discussed in Sec. 2.1.3, magnetic field solenoids, and connect to vacuum pumps. The glass tubes are held in place and sealed to the bells and the box port by a set of inflatable neoprene bladders which are pressurized to 65 psi. Including the bells, the experiment is 4.2 meters long and 1.5 meters wide. The entire experimental vessel rests on a rigid stainless steel support structure.

The base pressure in the vacuum vessel is maintained at 0.1 μTorr by an interlocked turbo-
Figure 2.1: The Line-tied Reconnection Experiment (LTRX)

Figure 2.2: CAD view of the Line-tied Reconnection Experiment
molecular pump and a cryogenic pump working in tandem. The turbo pump is backed by a dedicated scroll pump. A second scroll pump roughs out the chamber and provides differential pumping on probes. Rough vacuum is measured by a pair of thermocouple gauges; high vacuum is measured by a hot filament ionization gauge and a cold cathode gauge. Finally, a residual gas analyzer provides partial pressures of various gasses in the vacuum.

A box port halfway between the bells provides diagnostic access to the center of the plasma, pictured in Figure 2.3. The box port is constructed of acetal co-polymer plastic which maintains the insulating boundary condition. The box port seals to the tubes via inflatable bladders. Quartz plate windows on each side and the bottom of the box port allow optical diagnostic access. A plastic plate with small vacuum feed-throughs covers the top of the box port and allows direct probe access to the center of the plasma.
2.1.2 External Magnetic Solenoids

The experimental volume is threaded by a magnetic field of up to 1300 G. The field is directed along the axial direction of the cylinder. This field is provided by a total of 8 solenoids, seen in Figure 2.2. The solenoids are engaged several seconds before the plasma discharge occurs, yielding a temporal decoupling of solenoid and plasma generated magnetic fields.

The solenoids are powered by three direct-current power supplies controlled by a computer to analog interface. The solenoids and D.C. power supplies are cooled by a deionized water cooling loop. The deionized water exchanges heat with campus-provided chilled water. While the separate control of the solenoids allows for axial variation in the guide field, all data in this dissertation are from plasmas with uniform guide field. The magnetic ripple in this configuration is less than 4% between solenoids.

2.1.3 Plasma Generation and Current Control

The plasma is created by an array electrostatic washer guns originally design for current injection studies on the Madison Symmetric Torus [84, 85]. These guns, drawn in Figure 2.4, are capable of creating high density plasmas and independently driving high current densities. During, gas is puffed into the back of the gun. A few milliseconds later, a pulse forming network (PFN) power supply (sketched in Figure 2.5) creates a 100 V, 1 kA arc in the hollow tube between the gun cathode and the gun anode. Plasma from this arc then flows out the front of the gun at $v_z = \ldots$
$0.2c_s - 0.4c_s = 7 - 15 \text{ km/s}$, filling the experimental volume. The entire gun array is then biased relative to an anode at the other end of the experiment to drive plasma current.

A simplified drawing of the system is included in Figure 2.5. To maintain a constant equilibrium over the course of a discharge, the device uses a pulse-width modulation (PWM) system to control plasma current. The PWM system starts by firing an silicon controlled rectifier (SCR) and a gate turn-off thyristor (GTO) connected to a high voltage capacitor bank. This causes the current in the filtering inductor to quickly rise. Once the current reaches the pre-programmed level, the GTO switches to a lower voltage capacitor bank, causing the current to drop. Once the current drops below the set level, the GTO switches back to a medium voltage capacitor bank, increasing the current. This cycle continues throughout the discharge to drive a nearly constant plasma current with small (<5%) ripple.

A typical plasma discharge is presented in Figure 2.6. At $t=0$, a puff valve injects gas into the gun. At $t=2 \text{ ms}$, the PFN power supply engages, creating a plasma arc in the barrel of the gun. At $t=4 \text{ ms}$, the external biasing power supply engages, and plasma current ramps. From $t=6 \text{ ms}$ to $t=11 \text{ ms}$ the GTO switches between low and high voltage capacitor banks to maintain a constant current level. Finally, at $t=11 \text{ ms}$ the bias system shuts down and the plasma slowly decays as the PFN power supply turns off.

Eighteen plasma guns are packed in a hexagonal array at the cathode end of the experiment. This cathode is seen in Figure 2.8. The outer ring of twelve guns are powered solely by pulse forming networks. This means they can provide plasma density at large radii, but cannot drive plasma current. They were not used in this dissertation. The inner ring of six guns are independently biased to drive plasma current. While the plasma current is sourced at 6 discrete locations, they are observed to be azimuthally merged for the majority of the plasma column. This merging process is explored in Chapter 3. The center gun of the array was moved to the other end of the device for zero-net current experiments which are described in detail in Chapter 4.

Due to the discrete nature of the plasma guns, the line-tying condition is firmly satisfied at the cathodes [77, 60]. Thick copper anodes at the end of the device collect current emitted by the guns.
Figure 2.5: A simplified drawing of the gun power and biasing system. Image from Paz-Soldan [35].
Figure 2.6: Example traces from the plasma Guns.
Figure 2.7: The cathode, 18 plasma guns in a hexagonal mesh, and the central anode. The quartz shield covering the anode is not shown.

The anodes have a resistive dissipation time of $\tau \approx 200$ ms, indicating that magnetic fluctuations cannot penetrate the anodes on the timescales of a shot [79]. The combination of discrete guns and highly conducting anodes provides line-tying at both ends of the device.

2.1.4 Data Acquisition and Control System

When this dissertation began the laboratory was controlled by a Linux PC which used IDL scripts and custom GPIB drivers to control a large number of CAMAC based digitizers and timing units. Due to the age of the CAMAC system, the data acquisition system failed on approximately 2-3% of plasma discharges. Additionally, individual digitizers in the system failed on a regular basis and had to be replaced by an ever dwindling supply of surplus cards. To counter these problems and increase the reliability of the experimental systems, a new control and digitization scheme was implemented at the beginning of this dissertation.

Laboratory operation is controlled jointly by two computers. The first, a PC running MS Windows 7, controls the bias capacitor banks, solenoid power supplies, and lab cooling via a custom
NI LabVIEW control program and NI FieldPoint hardware. The second computer, a Dell Server running Centos Linux, communicates with digitizers and timing circuitry, stores data, and provides interactive plotting and monitoring of signals. Control on the Linux server is achieved via Python programs written as part of this dissertation.

Laboratory timing is controlled by a D-Taq AO32 analog/digital output card. Upon receiving a software trigger from the Linux Server, this card executes a series of pre-programmed digital pulses which trigger various parts of the experimental shot.

All experimental signals are digitized with D-Taq brand ACQ196 digitizers. Five 96 channel cards digitize a total of 480 channels of -10 V to 10 V, 16-bit, 500 ksp data. The D-Taq digitizers then upload data directly to an MDSplus [86] database running on the Linux server. With this system, each discharge of the plasma generates 11.5 megabytes of 16 bit data. The experimental campaigns described in the latter chapters each involve several gigabytes of experimental data.

2.2 Diagnostics

2.2.1 Segmented Anode

The linear nature of the device allows spatial and temporal measurement of currents in the device. Arc and bias currents from the guns are measured with shunt resistors. Currents to the end bells are measured via Rogowski coils. The segmented copper anode, pictured in Figure 2.8, collects currents from the guns at set positions. These currents are measured with Rogowski coils. The anode consists of a central disk and two concentric rings which measure the current at outer radii 2.3 cm, 5.2 cm, and 8.1 cm. Copper plates behind the rings catch any plasma flowing between the gaps. The discrete rings allow a rough current profile measurement at all times in the shot. The current measurement can be used to calculate the evolution of the safety factor profile \( q(r) \) at all times in the discharge.
2.2.2 External Magnetic Array

The radial structure of the magnetic field at the boundary of the experiment is measured continuously by an array of 80 pickup loops mounted just outside the vacuum vessel. The loops are 10 turns of copper wire printed on two flexible Kapton sheets. The coils are 6.5 cm x 12.5 cm and have an effective area of 730 cm². Sets of forty coil arrays are then wrapped around the vacuum vessel on either side of the box port. The radial array can measure magnetic modes of azimuthal wave number \( m < 4 \) and axial wave number \( .04 \text{ cm}^{-1} < k_z < 1 \text{ cm}^{-1} \). Signals from the coils are amplified and integrated by a set of Stirling Scientific Analog Integrators and then digitized.

2.2.3 Internal Magnetic Array

An array of magnetic probes was built during this dissertation to measure the internal structure of experimental plasmas in real time. 288 probes measure all three components of magnetic field on a 20 cm x 15 cm cross section at the axial mid-plane of the experiment. Signals from the array are amplified and integrated by a set of custom-designed analog integration circuits (detailed
A set of custom wound bobbins act as pickup loops for the probe. The bobbin is drawn in Figure 2.9 and a scale picture of the bobbin is shown in Figure 2.10. The bobbins were constructed and wound by Syrma Technologies. The coils are wound with 44 gauge copper wire to 200 turns. They have an effective area $NA \approx 2 \text{ cm}^2$. The tabs on either side of the main bobbin were built for ease of manufacturing and are cut off with small wire cutters before assembling each probe.
The probes are housed and aligned by a small plastic jig. Each probe is glued into the jig and wires come out on one end. The probe is then wrapped around with aluminized Mylar and inserted into a quartz tube that has been melted shut on one end. The open end of the quartz tube is then sealed with vacuum-safe epoxy.

The final array is comprised of 7 probes as shown in Figure 2.11. Each probe is inserted into the vacuum vessel through a sliding double o-ring seal on the box port. To minimize wasted channels, probes near the edge have fewer coils than probes at the center such that all probes are located within 3.5” of the plasma center (the array has roughly circular coverage). The array has 288 coils with 1.1 cm resolution in the Y direction and 2.5 cm resolution in the X direction. This array allows us to create magnetic movies of plasma activity as a function of time in the shot.

2.2.4 Optical Diagnostics

Three optical diagnostics were developed and implemented over the course of this dissertation. The simplest, a self-contained survey spectrometer built by Ocean Optics, allows qualitative assessment
of impurities in the plasma. When absolutely calibrated, this spectrometer can be used to estimate an ionization fraction of 90-95% [89].

The second optical diagnostic is a 1.5 m Czerny-Turner Spectrometer. The spectrometer was built in the 1960’s and originally designed to measure the amplitude of individual emission lines with a set of photo-multiplier tubes (PMTs). The spectrometer was retrofitted with a linear charge coupled device detector to measure the shapes of line emission from the plasma. In particular, the linear stark effect due to electron-neutral collisions in hydrogen can be used to measure electron density. The Lorentzian broadening of hydrogen-β emission is only weakly dependent on temperature and provides a volume averaged measurement of plasma density. The theory of the Czerny-Turner Spectrometer is discussed in Appendix C.

Finally, a laser interferometer was designed and built as an absolute measurement of plasma density. Unfortunately, vibration and sound issues during plasma shots introduced irremovable noise. These issues proved insurmountable and the the interferometer never produced usable density data. For reference and future edification, the design and theory of the laser interferometer is outlined in Appendix D.

2.2.5 Scanning Probes

At the beginning of this dissertation, an axially inserted scanning probe was routinely used to measure internal profiles of plasma density, temperature, and magnetic field [75]. Lengthening of the experiment and installation of the internal probe array prevented the use of this probe drive system for the majority of the data in this dissertation.

Just before the lengthening of the machine, a specialized scanning probe was used to measure the azimuthal merger of the 6 flux ropes. This was performed by installing a ring of 14 B probes measuring $B_\theta$ just outside of the plasma column. This probe, shown in Figure 2.12 measures the $m \leq 6$ structure of the magnetic field as a function of distance from the guns. These results will be discussed in Chapter 3.
Figure 2.12: The Ring Probe in the vacuum vessel. The probe was later upgraded from 12 to 14 coils to measure the $m=6$ component of $B_\theta$. 
2.3 Discussion

The extensive diagnostics and modular, independently controlled plasma source of the Line-tied Reconnection Experiment allow detailed time and space-resolved measurements of the line-tied screw pinch with complicated equilibria. The flexibility in the source allows the testing of equilibria that cannot be directly measured in astrophysical systems, while the high density diagnostics allow direct measurement of the complicated dynamics missed in other experiments that use shot-to-shot averaging [77, 68, 60, 79, 90, 31].

Typical plasma parameters in the device are shown in Table 2.1. It should be noted that the experiment is relatively cold due to the lack of confinement in the axial direction. Lundquist numbers are typically a few hundred, much lower than observed in astrophysical phenomenon. Additionally, the ion skin depth is large, typically 20-40% of the plasma radius. This means that experimental results may not directly compare to astrophysical observations. Experimental parameters do, however, match published simulations of astrophysical phenomenon.
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Formula</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Plasma length</td>
<td>$L$</td>
<td></td>
<td>2.08 m</td>
</tr>
<tr>
<td>Plasma radius</td>
<td>$a$</td>
<td></td>
<td>5 cm</td>
</tr>
<tr>
<td>Plasma duration</td>
<td>$\tau$</td>
<td></td>
<td>10 ms</td>
</tr>
<tr>
<td>Axial Field</td>
<td>$B_z$</td>
<td></td>
<td>1200 G</td>
</tr>
<tr>
<td>Plasma Current</td>
<td>$I_p$</td>
<td></td>
<td>1-6 kA</td>
</tr>
<tr>
<td>Electron Density</td>
<td>$n_e$</td>
<td></td>
<td>1E14 cm$^{-3}$</td>
</tr>
<tr>
<td>Elec. Temperature</td>
<td>$T_e$</td>
<td></td>
<td>4 eV</td>
</tr>
<tr>
<td>Spitzer Resistivity</td>
<td>$10^{-4} \log \left( \Lambda \right) T_e^{-\frac{3}{2}}$</td>
<td>$\eta$</td>
<td>65 $\mu \Omega m$</td>
</tr>
<tr>
<td>Elec. Thermal Speed</td>
<td>$(2k_b T_e / m_e)^{\frac{1}{2}}$</td>
<td>$v_{Te}$</td>
<td>840 km/s</td>
</tr>
<tr>
<td>Sound Speed</td>
<td>$(\gamma k_b T_e / m_i)^{\frac{1}{2}}$</td>
<td>$C_s$</td>
<td>25 km/s</td>
</tr>
<tr>
<td>Alfvén Speed</td>
<td>$B_z / (n_i m_i \mu_0)^{\frac{1}{2}}$</td>
<td>$V_A$</td>
<td>214 km/s</td>
</tr>
<tr>
<td>Mach Number</td>
<td>$v_z / C_s$</td>
<td>$M$</td>
<td>0.3</td>
</tr>
<tr>
<td>Alfvén Time</td>
<td>$L / V_A$</td>
<td>$\tau_A$</td>
<td>10 $\mu$s</td>
</tr>
<tr>
<td>Resistive Diffusion Time</td>
<td>$a^2 \mu_0 / \eta$</td>
<td>$\tau_{res}$</td>
<td>48 $\mu$s</td>
</tr>
<tr>
<td>Energy Confinement Time</td>
<td>$W / P_{Ohmic}$</td>
<td>$\tau_E$</td>
<td>10 $\mu$s</td>
</tr>
<tr>
<td>Lundquist Number</td>
<td>$\mu_0 a V_A / \eta$</td>
<td>$S$</td>
<td>200</td>
</tr>
<tr>
<td>Plasma Beta</td>
<td>$2 \mu_0 &lt; p &gt; / B^2$</td>
<td>$\beta$</td>
<td>2 %</td>
</tr>
<tr>
<td>Volumetric Ohmic Heating</td>
<td>$\int \eta J^2 dV$</td>
<td>$P_{Ohmic}$</td>
<td>1 MW</td>
</tr>
<tr>
<td>Ion Mean Free Path (MFP)</td>
<td>$(n_i \sigma_i)^{-1}$</td>
<td>$\lambda_i$</td>
<td>3 $\mu$m</td>
</tr>
<tr>
<td>Electron MFP</td>
<td>$(n_e \sigma_e)^{-1}$</td>
<td>$\lambda_e$</td>
<td>150 $\mu$m</td>
</tr>
<tr>
<td>Ion Skin Depth</td>
<td>$c / \omega_{ci}$</td>
<td>$\delta_i$</td>
<td>18 mm</td>
</tr>
<tr>
<td>Elec. Skin Depth</td>
<td>$c / \omega_{ce}$</td>
<td>$\delta_e$</td>
<td>0.4 mm</td>
</tr>
<tr>
<td>Elec. Larmor Radius</td>
<td>$v_{Te} / \omega_{ce}$</td>
<td>$\rho_e$</td>
<td>0.01 mm</td>
</tr>
</tbody>
</table>

Table 2.1: Table of plasma parameters for a typical device discharge. $k_b$ is Boltzmann’s constant, $\gamma$ is the adiabatic index, and $\Lambda$ is the coulomb logarithm.
Chapter 3

Azimuthal Merger of Individual Flux Ropes

Currents are injected into the experiment at six unique locations. These six flux ropes merge in a time-independent fashion to form azimuthally uniform plasmas for the majority of the length of the device. A neutral collision-based model for the diffusion of axial current is reviewed, along with a topological model based on magnetic shear. Together, these models semi-quantitatively explain internal measurements of plasma currents and justify the one-dimensional equilibrium assumption.

3.1 Introduction

The theory of screw pinch stability assumes azimuthally uniform current profiles. Plasmas in the device, however are created with six discrete sources. Experiments have observed a rich set of dynamics when two or three flux ropes interact, including instability, rotation around a common center of mass [73], magnetic reconnection and merger [61, 62, 63], and even bouncing [66]. The interaction of two and three flux ropes is studied in Chapter 6. As seen in other experiments, these interactions are dynamic.

Interestingly, the merger of 6 flux ropes in the experiment appears to be a smoother, less time-dependent process than the merger of two and three flux ropes. To explain this smooth, steady state merger, Paz-Soldan [35] proposed a advection-diffusion model and a topological azimuthal shear model. These models will be compared to recent data and discussed.
The outline of this chapter follows: Section 3.2 provides a summary of the model proposed by Paz-Soldan [35], Chapter 4. Section 3.3 presents a topological explanation of the azimuthal merger of currents. Section 3.4 presents z-resolved measurements of the m=6 current structure, while Section 3.5 presents internal, 2D measurements of current merger at one axial location. Section 3.6 summarizes the current understand of current merger in the device. This Chapter does not seek to provide a full physical description of the current merger process. Instead, it seeks to illuminate the current understanding of the process and present all relevant data.

3.2 Overview of Diffusion Advection Model

The advection-diffusion model of Paz-Soldan [35] begins with the resistive Ohm’s law:

$$E + v \times B = \eta J$$  \(3.1\)

where $\eta$ is the bulk plasma resistivity. We assume the azimuthal flow is nearly $v_{E \times B}$, except for a small offset parameter $\varepsilon$ ($\ll 1$). Thus $v = v_0 \hat{z} + (1 - \varepsilon) v_{E \times B}$, where $v_0$ is a constant. Inserting this definition of $\vec{V}$ into Eq. 3.1 yields a modified Ohm’s law:

$$E + (v' \times B) = \eta' J$$  \(3.2\)

where $v' = v_0 / \varepsilon$ and $\eta' = \frac{\eta}{\varepsilon} \left( (\varepsilon - 1) \frac{J}{T} + 1 \right)$ are modified flow speeds and resistivities. Magnetic probe measurements [79] show $J_\parallel \approx J$, thus we simplify $\eta' = \eta$.

Taking the curl of Eq. 3.2 twice yields:

$$\mu_0 \frac{\partial J}{\partial t} = \nabla \times \nabla \times v' \times B + \eta \nabla^2 J$$  \(3.3\)

Assuming time independence, spatially uniform $v'$, azimuthal symmetry, and a long thin approx-
Figure 3.1: Scanning B-dot probe measurements of axial current ($J_z$) profiles at different axial locations for plasmas created by 6 guns. Image from Paz-Soldan [35].

With a delta function for $J_z(z = 0)$, the $\hat{z}$ component of Eqn. 3.3 reduces to:

$$v_0 \frac{\partial J_z}{\partial z} = \epsilon \frac{\eta}{\mu_0} \nabla_\perp^2 J_z$$

(3.4)

With a delta function for $J_z(z = 0)$, the solution to Eqn. 3.4 is, in Cartesian coordinates:

$$J_z(r, z) = \sum_{i=1}^{6} A \exp\left(-\frac{(x-x_i)^2 + (y-y_i)^2}{4\epsilon \eta z/\mu_0 v_0}\right)$$

(3.5)

Where $\{x_i, y_i\}$ are the locations of the guns and $A$ depends on the total injected current. Thus, measurements taken at discrete axial locations are fit to solutions of these equations in Cartesian coordinates.

Figure 3.1 presents measurements of the axial current at four different axial locations. Using measured values of $v_0$ and $\eta$, we can fit Eqn. 3.5 to the data to find $A$ and $\epsilon$. This analysis gives $\epsilon \approx 1\% - 2\%$. The physical mechanism for the slowing of $\mathbf{E} \times \mathbf{B}$ drifts isn’t fully understood, but likely arises from neutral drag. Including neutral drag in 1.2, it can be shown [35] that the neutral
density $n_n$ is:

$$n_n = \left(\frac{\varepsilon}{1 - \varepsilon}\right) \frac{B_z^2}{\rho \eta_\perp \sigma v_{Ti}}$$

where the ion neutral collision frequency is $\nu_{in} = n_n \sigma v_{Ti}$. Using $\varepsilon$ from the current fit, this indicates an ionization fraction of 90\% – 95\%, consistent with optical measurements of neutral hydrogen emission [89]. The cross-field current transport is controlled by the Pedersen current [91].

### 3.3 Azimuthal Current Merger

A secondary mechanism for the merger of flux ropes involves a topological argument. The safety factor in the experiment is non-monotonic, meaning that field lines have different pitch angles at different radii in the plasma. Assuming $\mathbf{J} || \mathbf{B}$ (which is approximately correct with plasma $\beta =
Table 3.1: The measured distance for $m = 6$ merger, in cm versus current (rows) and guide field (columns). The flux ropes merge faster with more current and slower with more field, as expected.

<table>
<thead>
<tr>
<th>Current (kA)</th>
<th>500 G</th>
<th>550 G</th>
<th>600 G</th>
<th>650 G</th>
<th>700 G</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.5 kA</td>
<td>50</td>
<td>55</td>
<td>65</td>
<td>75</td>
<td>80</td>
</tr>
<tr>
<td>2.25 kA</td>
<td>40</td>
<td>45</td>
<td>50</td>
<td>60</td>
<td>65</td>
</tr>
<tr>
<td>3 kA</td>
<td>25</td>
<td>30</td>
<td>35</td>
<td>40</td>
<td>45</td>
</tr>
<tr>
<td>3.75 kA</td>
<td>15</td>
<td>20</td>
<td>20</td>
<td>25</td>
<td>30</td>
</tr>
<tr>
<td>4.5 kA</td>
<td>15</td>
<td>15</td>
<td>15</td>
<td>20</td>
<td>20</td>
</tr>
</tbody>
</table>

1% − 2%), the initially circular flux ropes are sheared azimuthally as they travel along the magnetic field. This happens as as currents at different cylindrical radii travel along field lines of different pitch, rotating through different azimuthal angles in a given axial space. This process is illustrated in Figure 3.2. Since this process depends on the pitch profile of magnetic field lines, it will vary with plasma current and applied guide field. The length should be longer with higher guide fields and shorter with higher plasma currents. This relation is explored more in the next section.

### 3.4 Ring Probe Measurements of Azimuthal Merger

To measured the steady state merger of flux ropes, a scanning probe was built. This probe, described in Section 2.2.5, uses 13 $\hat{B}$ coils to measure the spatial structure of $B_\theta$ at one radial location. This probe can measure the $m = 6$ component of $B_\theta$ created by the ring of 6 plasma guns. We use the $m = 6$ component as a proxy for the separateness of the individual flux ropes, meaning the flux ropes are merged when $B_{\theta,m=6} \rightarrow 0$. This is shown in Figure 3.3. $B_{\theta,m=6}$ is large near the guns and deceases to zero away from the guns. From this, we infer that the currents have merged azimuthally. A summary of the inferred merging lengths is displayed in Table 3.1. The lengths increase with guide field and decrease with plasma current, as expected by the magnetic shear model in the previous section.
Figure 3.3: This plot shows the \( m = 6 \) component of the magnetic field as a function of distance from the guns for a variety of guide field strengths. These plasmas used the ring of 6 guns and a total of 3 kA of plasma current.

### 3.5 Internal Array Measurements of Azimuthal Merger

The addition of the Internal Array, described in Section 2.2.3, allows direct merger measurement at \( z=104 \) cm. The flux ropes from the guns merge azimuthally before reaching the probe array for the vast majority of plasma parameters. This is seen in Figure 3.4 for a six gun plasma with high field (\( B=1300 \) G) and low current (\( I=750 \) A). The flux ropes merge azimuthally before the axial mid-plane of the experiment for almost all experimental conditions, the exception being the two and three flux rope plasmas studied in Chapter 6.
Figure 3.4: Current density from a six gun plasma from the internal array at \( z = 104 \) cm. This plasma has \( B = 1300 \) G and \( I = 750 \) A. Within the resolution of the probe array, the six flux ropes are azimuthally merged.

### 3.6 Discussion

The advection diffusion model and the topological merger model provide semi-quantitative explanations for the observed time-independent merger of the six flux ropes injected into the device. These models agree with measured plasma parameters, but are too simplified to explain the full 3D merger of the flux ropes. Nevertheless, they provide physical intuition toward the merger time independent merger of flux ropes. Results show that the equilibrium is approximately one dimensional for the majority of the experimental volume. These measurements justify assuming azimuthal symmetry when calculating screw pinch stability.
Chapter 4

The Zero-net Current Screw Pinch

This Chapter reports the first experimental investigation into a line-tied plasma with a reversed current profile. Discrete current sources create a cylindrical plasma equilibrium with an axial field and zero net current. Detailed magnetic measurements show that an internal $m = 1$ mode with no external character grows exponentially. The nonlinear evolution of the mode drives 3D reconnection events that reorganize the plasma equilibrium. The plasma exhibits dynamic fluctuations and reconnection events on a range of scales. These data are consistent with recent simulations of coronal loops and the nanoflare coronal heating mechanism. Brookhart et al. [92] is a shorter version of this Chapter.

4.1 Introduction

Current-driven magnetohydrodynamic (MHD) instabilities play an important role in many laboratory and astrophysical plasmas. The kink mode occurs when the plasma current twists magnetic field lines until they bend perpendicular to the field. In fusion experiments such as tokamaks, the external kink mode [6, 5] can grow rapidly and disrupt plasma operation, while the internal kink mode [7] can drive magnetic reconnection in the sawtooth cycle [8, 9]. Similarly, kink instabilities in coronal loops could contribute to the bursty magnetic energy release typical of solar flares [42].

In contrast to toroidal fusion devices, coronal loops are “line-tied” at the solar surface by the sharp transition from the magnetically dominated corona to the pressure dominated photosphere. Gold and Hoyle [45] proposed a solar flare model where magnetic field in a coronal loop is slowly
twisted by motions in the Sun’s photosphere. Localized twisting motions in the photosphere generate a coaxial current structure with zero net current. In cylindrical coordinates, \( J_\parallel > 0 \) at small \( r \) and \( J_\parallel < 0 \) at large \( r \).

Stability analysis for line-tied plasmas is more complicated than the analysis of periodic plasmas. An instability with a single wave number, appropriate for periodic plasmas, cannot satisfy line-tied boundary conditions. Instead, multiple modes with different wave numbers or three-dimensional simulations must be considered. Most numerical studies of the stability of the zero net current coronal loop use a straight, finite-length cylindrical approximation to toroidal geometry of coronal loops. [48, 49, 52, 53, 54, 55, 20]. In these simulations, the zero net current coronal loop is unstable to an exponentially growing internal kink mode when the twist in the magnetic field (\( \Phi(r) \)) is sufficiently large. Equivalently, the internal kink mode is unstable when the safety factor

\[
q(r) = \frac{2\pi \Phi(r)}{L B_\theta} < q_{\text{crit}}
\]

(4.1)

where \( L \) is the length of the cylinder, \( r \) is the radial location, and \( B_\theta \) is the field generated by plasma current. The exact value of \( q_{\text{crit}} \) depends on the twist profile applied to the loop, but most zero net current equilibria are unstable at \( q_{\text{crit}} = 0.5 - 0.6 \).

While line-tied plasmas cannot form the current singularities present in toroidal geometries, early nonlinear simulations showed that the line-tied internal kink mode forms strong current sheets in the zero net current loop [52, 53, 54, 55]. These current sheets then reconnect, dissipating up to 90% of the nonpotential magnetic energy in the loop. In recent higher resolution simulations, the kink-driven current sheet fragments into multiple reconnection sites and the system transitions into a self organized state of magnetically driven turbulence [56, 58].

To date, no experiments have created or studied the zero net current equilibrium. In tokamaks, experiments were performed attempting to create transient reversed current profiles using sources of noninductive current drive [33, 36, 38, 39]. Plasma self-organization clamped the current profile to zero on axis, preventing a reversed current equilibrium. Line-tied experiments have focused exclusively on plasmas with unidirectional current [77, 60, 78, 79, 81, 90].
This Chapter presents the first experimental investigations into the stability of the zero net current equilibrium. Using a cylindrical, line-tied screw pinch experiment, the zero net current equilibrium is created for the first time in a laboratory. The equilibrium exhibits kink-like behavior, reconnection events, and dynamic, turbulence-like fluctuations. Comparisons to simulations are presented.

4.2 Experimental Setup

To create a zero-net current equilibrium, six guns produce a ring of negative plasma current at \( r = 3.63 \) cm while a seventh gun, located on the other end of the experiment, provides positive current at the center of the device (see Fig. 4.1). The central bias system is resistively floated to prevent arcs at either end of the experiment. Quartz tubes surround the central gun and anode and extend 4 inches into the cylinder. These tubes provide additional insulation to prevent arcs.

Alfvén wavelengths can propagate in the cylinder at frequencies less than the ion cyclotron frequency. Using \( \omega_A = \omega_{ci} \), approximately 15 Alfvén wavelengths fit in the cylinder, providing ample space for Alfvénic turbulence. The Alfvén crossing time \( \tau_A = L/v_A \approx 10 \) \( \mu \)s is much shorter than the plasma discharge. The Lundquist numbers for these plasmas are \( S_\parallel = \mu_0 B_\parallel a/\eta \sqrt{\rho \mu_0} \approx 120 \) and \( S_\perp = \mu_0 B_\theta a/\eta \sqrt{\rho \mu_0} \approx 10 \). While experimental Lundquist numbers are much lower than coronal values, they are comparable to Lundquist numbers in published nonlinear simulations. The ion skin depth \( \delta_i/a = c/\omega_{pi} a \approx .4 \), where \( a = 5 \) cm is the radial size of the plasma, indicating two
Figure 4.2: An example discharge. (a) The current from the center gun, the outer guns, and the total plasma current. The plasma is created at $t = 2 \text{ ms}$ and the biasing power supplies are engaged at $t = 4 \text{ ms}$. The plasma current is slowly increased over the course of the shot to prevent disruptions to the biasing power supplies. (b) The solid line is a trace from an internal magnetic probe located at $z = 1.04 \text{ m}$, $r = 0.5 \text{ cm}$; the fast fluctuations are due to internal plasma instabilities. The dashed line is a trace from an external probe located at $z = 0.96 \text{ m}$, $r = 10 \text{ cm}$; the internal instabilities do not appear on external diagnostics.

Fluid effects may be important.

4.3 Results

An example plasma discharge is presented in Fig. 4.2. Part (a) plots current time traces from the central gun and the outer ring of six guns. The total current in the plasma, indicated by the light gray line, is close to zero. Example magnetic traces, in part (b), show large fluctuations inside the plasma but no fluctuations at the vacuum vessel. Time averaged spatial profiles of the magnetic field are charted in Fig. 4.3. A clear current reversal is present at $r = 3 \text{ cm}$. $B_\theta = 0$ at $r = 9 \text{ cm}$; thus, Ampere’s law indicates zero net current in the plasma. For this equilibrium, the safety factor $q$ is minimal on axis and $q \to \infty$ at large radius.

To illustrate the nature of the magnetic fluctuations in Fig. 4.2 (b), we have explored the dynamics of the large events. An Example is shown in Fig. 4.4 (a). Large mode activity occurs in the center of the plasma but the external magnetic array exhibits no $m = 1$ activity above the
Figure 4.3: An example zero-net Current equilibrium. (a) The measured $\Delta B_z$, $B_\theta$, and pressure profiles. Note that $B_\theta = 0$ at the edge of the vessel. The plasma is roughly isothermal, pressure gradients are due to decreased density at the plasma edge. (b) The current density with the reversal at 3 cm and the steep $q$ profile.

Figure 4.4: A reconnection event. The solid black line in (a) is the $m = 1$ component of $B_r(r = 0)$. Exponentials are fit for $-20 < t < -8 \mu s$ and $-20 < t < 0 \mu s$ to find mode growth rates. The $m = 1$ component of the external magnetic array (dotted line in (a)) is below the noise floor. (b) shows the sharp increase in safety factor at the peak of the $m = 1$ mode.
Figure 4.5: (a) The time-averaged current density. While the plasma current is sourced at discrete locations, currents have merged azimuthally prior to reaching the probe array. (b) The current perturbation at the peak of the $m = 1$ activity. The $m = 1$ perturbation is clear, as is evidence for a current sheet at $(x, y) = (-1, 4) \text{ cm}$. Noise floor. The mode exhibits exponential growth, but flattens from -8 to -4 $\mu$s. This may be noise or nonlinear physics, so two exponentials are fit to find the growth rate. Both growth rates $\gamma a/v_A = 0.027 \pm 0.006$ and $\gamma a/v_A = 0.036 \pm 0.005$ are consistent with the linear growth rate of the ideal MHD internal kink mode in plasmas with zero-net-current equilibria [49, 52, 20].

Fig. 4.5 (a) presents a two-dimensional measurement of the plasma equilibrium. While the plasma current is sourced from discrete guns, current is azimuthally smoothed at the center of the experiment. The $m = 1$ eigenstructure of the mode is clearly visible in Fig. 4.5 (b). Additionally, a strong current sheet occurs at $(x, y) = (-1, 4) \text{ cm}$. This spike in the current density, lasting for $\approx 6 \mu$s, is approximately 1.1–1.5 cm thick and 6 ± 1 cm long at the axial location of the array. Less
Figure 4.6: The current profile 20 $\mu$s before (black) and after (gray) the crash shown in Fig. 4.4. (a) shows a fit to the internal magnetic data measured 1.04 m from the guns. Notice how the central current density decreases by a factor of 2. Error on the fit is +/-2 A/cm$^2$. (b) shows the current profile measured at the anode (2.08 m from the guns) of the experiment. The drop in central current is much lower.

coherent, higher order structures are also routinely observed.

When the $m = 1$ mode reaches its peak amplitude, the safety factor at the center of the plasma begins to rise, as shown in Fig. 4.4 (b). This is consistent with the redistribution of current expected in a magnetic reconnection event. In total, approximately 40% of the nonpotential magnetic energy is dissipated by this event. Fig. 4.6 graphs the current profile measured by (a) the internal array, and (b) the segmented anode, 20 $\mu$s before and after the peak of the mode. There is a large drop in plasma current at the axis of the machine but very little change at the boundary of the experiment. This indicates three-dimensional reorganization of the equilibrium profile.

The growth rate, eigenstructure, and reconnection dynamics presented here are consistent with simulations of the ideal MHD internal kink mode in a zero net current plasma [52, 54, 55, 56, 58, 23]. Unfortunately, this event does not match expected stability criteria: $q = 1.9 \pm .2 \gg q_{\text{crit}} = 0.5 - 0.6$ [49, 52, 93, 20]. Resistivity can destabilize the line-tied kink mode but the effect is not strong enough to explain these data [17]. Similarly, linear NIMROD [94] simulations (Fig. 4.7, performed with experimental parameters and profiles) indicate that the Hall effect can destabilize
the kink by 5 – 10\%, and is also not enough to explain the data. An analysis of 2078 reconnection events in the data shows no critical instability conditions. These events display a wide range of safety factors \( q = 1.2 – 3 \), current sheet sizes \( L \leq 1 – 6 \text{ cm} \), and energy dissipation \( 10\% – 80\% \).

Instead of an internal kink mode, these events may be self-organized structures. An aggregated power spectrum in Fig. 4.8 shows a clear \( f^{-5/3} \) spectrum. Measurements of the wave number spectrum (Fig. 4.9) show a steeper wave number spectrum \( k_\perp^{-4\pm1} \). This fluctuation spectrum, combined with the large spread in reconnection dynamics we observe, indicates of a turbulence-like state in this dissipative system. The experiment exhibits fluctuations very early in the plasma discharge (as shown in Fig. 4.2 (b)) and may be seeded by fluctuations in the guns. Interestingly, these fluctuations seem to relax the equilibrium, preventing the equilibrium from reaching linear instability.

This picture of turbulent relaxation is qualitatively consistent with recent simulations by Rappazzo et al. [23]. In those simulations, magnetically driven turbulence stochastically dissipates...
Figure 4.8: The power spectrum of magnetic field calculated with the Morlet wavelet. The dashed line indicates $f^{-5/3}$, matching the data. Additionally, two coherent modes rise above the background spectrum. The lower frequency mode is the plasma rotation due to $\vec{E} \times \vec{B}$ drifts. The higher frequency mode might be drift-Alfvén activity, and is beyond the scope of this work.

Figure 4.9: The power spectrum of magnetic field vs wave number as measured by the probe array. Noise in the profile is due to limited spatial resolution.
magnetic energy at a variety of scales. These turbulent reconnection events relax the coronal loop, raising the safety factor and stabilizing the kink mode for the duration of the simulation. These turbulent reconnection events heat the corona, similar to the Parker model of nanoflares [43, 44].

### 4.4 Discussion

These results are the first demonstration of the zero-net-current equilibrium in the laboratory. These data are also the first unambiguous demonstration of internal modes driving three-dimensional reconnection in a line-tied plasma. Bergerson et al. [77] presented evidence of the internal kink mode in a line-tied screw pinch with a monotonic current profile, but no internal diagnostics were available for those experiments. The lack of internal data left some ambiguity on the internal and external nature of the observed mode [78].

Despite the fact that equilibria are approximately one dimensional, zero-net-current plasmas exhibit strong dynamics well below required parameters for instability. Fluctuations in the plasma guns may be a sub-critical drive for turbulence. Plasma fluctuations then feeds off the current drive, dissipating magnetic energy at a variety of scales.

Many reconnection events in this plasma are clearly self-organized structures. At this point it is difficult to determine if the larger $m = 1$ modes observed are the result of an internal kink with a modified stability criterion or if they are also self-organized turbulent structures. Reconnection events likely heat the plasma, but current temperature diagnostics are limited by sensitivity.

While the experimental boundary conditions match the corona, the current drive in the experiment may not react to reconnection events the same way that the corona does. This may change the statistical nature of turbulence in the simulation. The Hall effect is measurable in the experiment but negligible in the corona. This will also change the statistics of the turbulence.

Current experimental capabilities do not measure the three-dimensional topology of reconnection events. Additional probes at other $z$ locations would allow measurements of field lines and the three-dimensional structure of reconnection events. Electrostatic probe arrays co-located with
the magnetic probes would measured density and electric field fluctuations allowing more direct comparisons to theories of turbulence and magnetic reconnection.
Chapter 5

The Hollow Screw Pinch

Inspired by recent “current hole” experiments on tokamaks, this chapter presents measurements of line tied plasmas with hollow current boundary conditions. The plasma exhibits self-organized states at all parameters. At high guide field, this takes the form of a rotating hollow screw pinch. As guide field is lowered, the state bifurcates. The new state fills in the current profile and creates a large scale, rotating helical screw pinch with a flat current profile. The bifurcation occurs at a critical guide field, independent of plasma current. The data are compared to theories of pressure and current drive instabilities and waves.

5.1 Introduction

Studies of the screw pinch equilibrium traditionally focused on screw pinches with peaked central current. The reason for this is simple: confinement is best at the center of the screw pinch in both toroidal and linear devices. This means that plasma temperature, and thus conductivity, is typically higher the center of the pinch. Current then follows the path of least resistance, creating a centrally peaked screw pinch equilibrium.

More complicated screw pinch equilibria have been difficult to achieve in the lab, but theorists have long considered the stability of a screw pinch with a hollow current profile. For instance, Shafranov [6] and Kruskal and Tuck [5] calculated the stability of the external kink mode for a plasma with no volume current and a surface current at the plasmas edge, and Ryutov et al. [13]
revisited the surface current external kink mode for line tied systems. These models only consider the external stability of the kink mode, ignoring any dynamics that may be present inside the plasma.

The internal dynamics and structure of the hollow screw pinch was not fully considered until the early 2000’s when a number of fusion devices began experimenting with off-axis current drive by neutral beam, radio-frequency waves, and the bootstrap current. These off axis current drives induce a negative toroidal electric field on the axis of tokamaks, creating a region with no current at the center of the experiment. These “Current Hole” equilibria have been observed on JT-60U [33], JET [36, 37], ASDEX-U [38], and DIII-D [39].

In certain cases, off axis current drive was so strong that on-axis current should have been driven negative. Interestingly, all experiments performed to date show that the current “clamps” to zero, never becoming negative. This is interpreted as a self-organized system, where self-driven flows or the repetitive action of the resistive kink mode clamp the current on axis to zero [40].

This chapter presents experiments on the line-tied screw pinch with a hollow current equilibrium. The equilibrium is created and maintains it’s hollow nature through the discharge for certain plasma parameters. Instabilities are observed which collapse the hollow current profile at very large safety factors. The scaling of this modes is discussed.

It should be noted that that the hollow current screw pinch may play a role in astrophysical systems. It is, for example, easy to imaging a photospheric twisting profile (see Sec. 1.6) that creates a hollow current coronal loop. This system hasn’t been studied in an astrophysical context. Thus, this chapter will not elaborate on possible connections to astrophysical systems.

### 5.2 Experimental Setup

These experiments were performed on the Line-Tied Reconnection Experiment, described in Chapter 2. Hollow current plasmas were created with six washer guns, as illustrated in Figure 5.1. The hollow current profile is rigorously enforced at both ends of the experiment. At the cathode end
of the experiment, the hollow profile is set by the 6 plasma guns. At the anode end of the experiment, the electrode setup described in 2.1.3 prevents current from leaving the experiment at $r < 2.3$ cm, thus enforcing a hollow $J_z$ profile. From this we can inferred that current at the center of the experiment is indicative a three dimensional process.

The axial field $B_z$ ranged from 600 G to 1300 G by 50 G intervals while plasma current was varied in the set $I_p \in \{0.75, 1.125, 1.5, 1.875, 2.25, 3, 3.75, 4.5\}$ kA. Plasmas in this data set have density $n_e = .5 - 1 \times 10^{20}$ m$^{-3}$ and plasma beta $\beta = 2\mu_0 p / B^2 < 2\%$. Two plasma discharges were created at each $B_z$, $I_p$ pair for a total of 240 shots. In raw 16 bit format, this data set occupies 2.65 GB of disk space.

5.3 Results

An example discharge is presented in Figure 5.2 for a plasma with $B_z=1300$ G and $I_p = 750$ A. This discharge has the highest safety factor in the data set: $q$ is greater than 10 for the entire discharge. Despite this fact (and the inferred stability of current driven modes), the plasma exhibits strong fluctuations for the duration of the discharge. This is inconsistent with theories of the kink mode [14].
Figure 5.2: An example hollow current discharge with 1300 G guide field and 750 A plasma current. (a) shows the total current in the plasma, (b) is a time trace from a $B_\theta$ coil at the center of the experiment. Despite the high Guide Field and Low plasma current, the magnetic coils shows coherent oscillations.

### 5.3.1 Equilibrium Profiles

To explain this high safety factor fluctuations, we first turn to equilibrium profiles. Figure 5.3 shows the measured equilibrium for the $I_p=750$ A, $B_z=1300$ G plasma displayed in Figure 5.2. The plasma has a hollow pressure profile, peaking at $r=3$ cm, and a hollow current profile, peaking at $r=4$ cm. Safety factor is smallest at $r=4.2$ cm, rising for both smaller and larger $r$. Part (b) of the figure displays the current density profile in 2D at a single time point. This indicates that the plasma current is indeed hollow.

For completeness, equilibrium profiles for an $I_p=750$ A, $B_z=1000$ G plasma (Figure 5.4) and an $I_p=750$ A, $B_z=750$ G plasma (Figure 5.5). Between 1300 G and 1000 G there is little change in the observed plasma profiles. Currents profiles are slightly less hollow at 1000 G, and absolute pressures are slightly smaller, but overall the plasmas exhibit very similar equilibria. In contrast, the plasma at $B_z=750$ G exhibits a radically different equilibrium. The current profile is completely filled in and pressure profiles are much smaller. This process appears to be regulated by fluctuations in the plasma, as discussed in the next section.
Figure 5.3: (a) Equilibrium profiles of the Hollow Current Screw Pinch with $B_z = 1300$ G and $I_p = 750$ A. Profiles are calculated using the fitting method described in Appendix B and time averaged over 1 ms. They have uncertainty $\approx 10\%$. (b) Instantaneous 2D contour plot of current density, showing the hollow profile.
Figure 5.4: (a) Equilibrium profiles of the Hollow Current Screw Pinch with $B_z = 1000$ G and $I_p = 750$ A. The current density and pressure profiles have filled in slightly (b) Instantaneous 2D contour plot of current density.
Figure 5.5: (a) Equilibrium profiles of the Hollow Current Screw Pinch with $B_z=750 \, \text{G}$ and $I_p = 750 \, \text{A}$. The pressure and current profile have completely filled in. (b) Instantaneous 2D contour plot of current density, showing flat profile.
5.3.2 Stability and Fluctuations

To explore the nature of the fluctuations, Figure 5.6 presents spectograms for the three plasma discharges discussed in the previous section. All three plasmas have plasma current $I_p=750$ A and guide field is varied between shots. At $B_z=1300$ G, the spectogram shows a coherent magnetic oscillation at $f \approx 10$ kHz. As the guide field is lowered to 1000 G, the plasma becomes more chaotic, exhibiting bursty behavior at a variety of frequencies. Finally, when guide field is lowered even further to 750 G, a new coherent mode emerges. This mode is much slower ($f \approx 2$ kHz) and has a much larger amplitude than the mode at 1300 G. Once again, $q > 4$ for all of these plasmas, indicating that kink-like modes are stable.

I employ Principle Component Analysis [95] to find the eigenfunctions of these fluctuations. Using data from both the internal and external probe arrays we can plot the dominate eigenmodes of the each fluctuation spatially. Figure 5.7 shows the calculated fluctuation eigenfunctions for the plasma with 1300 G. The internal eigenfunction has an $m=1$ structure that rotates azimuthally. Internal fluctuations are localized to either side of the peak in equilibrium current density, and correspond to a simple (real) radial eigenfunction. This eigenfunction corresponds to a rigid shift in the current profile that rotates around the center of the experiments in the $\hat{\theta}$ direction.

The fluctuation eigenfunctions for the plasma with $B_z=1000$ G are not shown due to their similarity with Figure 5.7. The main difference between the 1000 G plasma and the 1300 G plasma is time dynamics. While the $m=1$ mode saturates and rotates in the 1300 G plasma, it appears in dynamic bursts in the 1000 G plasma. The origins of this dynamic behavior are unclear.

The eigenfunction for the 750 G plasma is shown in Figure 5.8. At this point the fluctuations have clearly changed. The external eigenfunction is still localized to the anode end of the experiment and rotates, but the internal eigenfunction has a much larger radial extend and cannot be described as a real radial eigenfunction. Instead, this mode must be two modes interacting with each other. This plasma is in a self-organized, three dimensional state that rotates $\hat{\theta}$ direction.
Figure 5.6: Spectograms calculated from the external magnetic data with the complex Morlet Wavelet. All three plasmas have $I_p = 750$ A. (a) shows a plasma with $B_z = 1300$ G, a coherent mode is evident at 10 kHz. (b) shows a plasma with $B_z = 1000$ G, the plasma has become magnetically chaotic. (c) shows a plasma with $B_z = 750$ G, a strong 3 kHz mode now dominates the plasma.
Figure 5.7: (a) Internal $J_z$ Eigenfunction measured by the probe array at $z=1.04$. This is an $m=1$ perturbation that is localized in $r$ and rotates around the center of the plasma. (b) External $B_r$ eigenfunction for the same mode, $z$ is measured from the guns, the anode is located at 208 cm. This shows that the fluctuations are $m=1$ and localized towards the anode. These fluctuations also rotate.

Figure 5.8: (a) Internal $J_z$ Eigenfunction measured by the probe array at $z=1.04$. This is an $m=1$ perturbation that extremely broad in $r$ and rotates around the center of the plasma. (b) External $B_r$ eigenfunction for the same mode, $z$ is measured from the guns, the anode is located at 208 cm. This shows that the fluctuations are $m=1$ and localized towards the anode. These fluctuations also rotate.
Figure 5.9: (a) External $B_r$ rms fluctuation levels weighted by plasma current. The fluctuations for different plasma currents are very similar as a function of $B$, except where $q < 1$. (b) safety factors for the shots. The majority of discharges have $q > 1$, but a few at high $I_p$ and low $B_z$ do cross the kink threshold. These show higher fluctuation levels.

5.3.3 Fluctuation Scaling

Finally, we consider the scaling of fluctuations across the entire data set. Figure 5.9 (a) shows the rms magnetic fluctuation levels of all the shots in the data set, weighted by plasma current. When corrected for plasma current, the fluctuations correlate very well across plasma discharges as a function of $z$. This is mostly independent of safety factor $q$ (Figure 5.9 (b)), except for a few shots where $q < 1$ that have slightly higher fluctuation levels. For all plasma currents, the fluctuation amplitudes grow when $B_z < 800-900$ G and settle into a rotating, helical structure as shown in Figure 5.8. The critical guide field for this self-organized structure is not well understood, but is explored in the Discussion.

This self organization can also be observed in the Complexity-Entropy map presented in Section 1.4.2. As shown in Figure 5.10, high and medium guide field plasmas are chaotic, matching deterministic chaos maps very well. As the guide field is lowered, plasma self-organized into a coherent structure at $B_z \approx 800$ G, independent of plasma current and safety factor. When this occurs,
Figure 5.10: The Complexity-Entropy Plane of all Hollow Current plasmas for \( n = 5 \). Each dot is the result from a single probe and dots are colored by the strength of the guide field. High guide field plasmas exhibit chaotic behavior. As guide field is lowered beyond approximately 800 G, plasmas self-organize into a coherent state and move toward the sine-wave portion of the complexity plane. This is true for all plasmas, regardless of current.

plasmas have much lower entropy, landing near sine wave functions on the complexity map. The physical mechanisms responsible for these modes are considered in the next section.

5.4 Discussion

This chapter presented line-tied hollow current plasmas intended to replicate self-organized behavior observed in toroidal device. The line-tied plasmas did not replicate the stable hollow-current profile observed in tokamaks. Instead, the line-tied plasmas self-organized into a rotating, helical and hollow current state. As the guide field was lowered, the plasma bifurcated into a rotating helical state with lower rotation frequency and a flat current profile. This bifurcation to depends on a critical guide field strength, independent of plasma current.

The localized eigenstructures in high guide field plasmas resemble both localized interchange modes and localized drift modes. Indeed, the profiles in Figure 5.3 mean that Suydam’s Criterion
[96] (necessary for the stability of a periodic screw pinch)

\[ rB_z \left( \frac{q'}{q} \right)^2 + 8p' > 0 \]  

(5.1)

is violated in a small region around \( r = 4.2 \) cm. In contrast, Hood [97] presented an extended Suydam-like criterion for the stability of the line-tied screw pinch. Using this criterion, the profiles presented in Figure 5.3 are stable to localized interchange modes. Since the plasma is clearly fluctuating, a line-tied interchange mode seems unlikely.

It is also possible that the observed fluctuations are due to the resistive drift instability. Drift waves travel with the electron diamagnetic drift velocity [98]

\[ v_{de} = \frac{kT_e}{en_eB^2} \nabla n_e \times B \]  

(5.2)

Based on the profiles in Figure 5.3, we can estimate that a drift wave would travel at 450 m/s. This yields a rotation frequency of 11 kHz at \( r = 4 \) cm, consistent with the observed data. Unfortunately, for the region in question (at \( r = 4.2 \) cm), the direction of \( \nabla n_e \) is \(-\hat{r}\) and the direction of \( B \) is primarily \(-\hat{z}\). Thus, drift wave propagation is in the \(-\hat{\theta}\) direction, inconsistent with measured mode rotation. Alternatively, the high field mode could be a drift wave unstable in the region \( r < 2 \) cm. This wave would travel in the \( \hat{\theta} \) direction, consistent with the observed mode, and could couple to larger radii.

A third option to explain these modes is the kink mode. The measured helical states are both right handed, consistent with the paramagnetic kink mode [99]. Additionally, the modes both “screw” into the anode of the experiment, consistent with theories of kink mode rotation in the presence of axial flow [14, 69]. Measured safety factors, however, are much too high for the mode to be an ideal kink mode. Imperfect line tying [14] and plasma resistivity [14, 17] can destabilize the kink, but neither effect is strong enough to explain the observed instabilities or the observed critical magnetic field.

While the mode responsible for the fluctuations is unclear, this is clearly a chaotically self-
organized state. In this sense, these results are consistent with theories of plasma relaxation in resistive MHD [100] and Hall MHD Khalzov et al. [101]. Fluctuations are initiated at high guide field by some undetermined mode. As guide field is lowered, these fluctuations become chaotic and self-organize into a preferred state.

This is the first experimental evidence of a self-organized relaxed state in a line-tied system. The constraints on this state (such as conservation of helicity [100]) are unclear. Three dimensional measures may elucidate the modes responsible for the initial fluctuations and the properties of the relaxed, dynamic state.
Chapter 6

The Interaction of Flux Ropes

Plasmas with fully three dimensional structure are created with two and three flux ropes. Equilibrium measurements show that the flux ropes twist around each other in three dimension. Flux ropes are chaotic at all parameters, exhibiting intermittent broadband fluctuations. The fluctuations are chaotic, indicating a turbulence-like process. At low safety factor, fluctuations may undergo an inverse cascade.

6.1 Introduction

The magnetic flux rope is the basic building block of magnetic systems in many astrophysical plasmas, including the Corona, the Solar Wind, and the Earth’s Magnetosphere [41]. These flux ropes are usually force free bundles of twisted magnetic field and current. Individual flux ropes closely resemble the screw pinch, and exhibit internal [77] and external [68, 80] kink modes that depend strongly on boundary conditions [14, 70, 71].

When multiple flux ropes interact with each other, more complicated phenomena can arise. The simplest of these multiple flux rope plasmas is the case where two parallel flux ropes exist side by side. In this two dimensional geometry, the flux ropes are mutually attracted by $\mathbf{J} \times \mathbf{B}$ forces and repelled by magnetic pressure. If the attractive forces are strong enough, magnet reconnection occurs between the two flux ropes and they merge [102, 59, 60].

The picture becomes significantly more complicated when three dimensional effects are con-
sidered. Sun et al. [66] showed that kinking flux ropes can bounce off of each other due to a magnetic tension restoring force. This is a fully three-dimensional process, leading to different collision, bouncing, and magnetic reconnection rates at different places along the flux ropes [62]. When three flux ropes are considered, reconnection and bouncing dynamics become even more complex [63].

Experiments on interacting flux ropes from Los Alamos National Laboratory [61, 66] and the University of California – Los Angeles [73, 62, 63] have used scanning probes and shot-to-shot reproducibility to build large three dimensional data set of interacting flux ropes. Unfortunately, this techniques necessitates the removal of any plasma behavior that is not reproducible. Using statistical techniques from chaos theory, Gekelman et al. [31] showed that flux rope experiments at UCLA regularly show complex and mathematically chaotic behavior under a range of plasma conditions. This chaotic behavior disappears when shots are averaged to create 3D datasets, so the authors could not elucidate the physical mechanisms responsible for chaos in the magnetic flux ropes.

In this chapter I present experiments on the interaction of two and three magnetic flux ropes from the Line-Tied Reconnection experiment. Two probe arrays with a total of 368 magnetic coils yield detailed 3D measurements of the flux ropes in a single shot. This allows direct measurement of flux rope behavior, including chaos, without the shot-to-shot averaging used in other experiments.

6.2 Experimental Setup

These experiments were performed on the Line-tied Reconnection Experiment, as described in Chapter 2. Two sets of experiments were performed. The first used two plasma guns to create two flux ropes separated by 7.26 cm as illustrated in Figure 6.3 (a). The second set of experiments used three plasma guns to create three flux ropes separated by 6.29 cm as illustrated in Figure 6.3 (b). Plasma currents in each flux rope were identical for each discharge. The current and background
Figure 6.1: An illustration of the plasma cathode. (a) shows which guns are used for the two flux rope experiments, and (b) shows which guns are used for the 3 flux rope experiments.

Field were varied from shot to shot. $I_{gin}$ ranged from 125 A to 750 A by 125 A intervals and $B_z$ ranged from 500 G to 1300 G by 50 G intervals. Two plasma discharges were created at each current/guide field setting, for a total of 204 plasma discharges in each data set. In raw 16-bit format, these data sets are 2.3 GB each.

The safety factor $q$ will be used to understand stability of these flux ropes. Using Ampere’s law, the safety factor in a flux rope can be written

$$q = \frac{4\pi^2 a^2 B_z}{\mu_0 LI}$$

where $a$ is the radius of the flux rope, $L$ is the length of the experiment, and $I$ is the current in each flux rope. Individual flux ropes are measured to have a radius $a \approx 2$ cm (see Figure 6.1). The data set ranges from $q = 6.3$ for $B_z=1300$ G, $I=125$ A flux ropes to $q = .4$ for $B_z=500$ G, $I=750$ A flux ropes. The kink mode is unstable at $q=1-2$, depending on line-tying boundary conditions [14]. Thus, the data includes flux ropes that should be fully stable and wildly unstable.
6.3 Results

Time traces from an example discharge are shown in Figure 6.2. Part (a) shows the current in each flux rope. Each flux rope carries approximately the same current at all points in the discharge. Figure 6.2 (b) shows the trace from a single magnetic coil located at the center of the experiment. Despite the fact that these flux ropes have \( q \approx 6 \), the magnetic coil measures large fluctuating magnetic fields, indicative of an instability or interaction between the flux ropes.

6.3.1 Equilibrium

Before investigating the origin of fluctuations in flux rope plasmas, we will consider the structure of the flux ropes. Figure 6.1 shows current density contours as measured by the internal probe array at \( z = 1.04 \) m for two and three flux rope plasmas with \( B_z = 1300 \) G and \( I = 125 \) A per rope. Individual flux ropes are clearly identifiable in the contour plots and align very well with the positions of the plasma sources at \( z = 0 \) m.

Principle Component Analysis (PCA) [95] is used to separate the low frequency equilibrium
components of the magnetic signals from various fluctuating modes. PCA works by finding the components of the signals with the highest correlations. It returns a series of time traces and eigenfunctions for the various modes in the data set, sorted by their statistical importance. It works as an unsupervised machine learning method to find the most important modes in the plasma.

The most energetic mode returned by PCA is, of course, the plasma equilibrium. This mode is approximately constant in time and reflects the average position of flux ropes throughout the discharge. A simple analysis of the equilibrium shows that the flux rope equilibria should twist into a helical structure at higher currents and lower guide fields. To explain this, consider the fact that the flux ropes are approximately force free \((\beta \approx 1\%)\), indicating \(\mathbf{J} \parallel \mathbf{B}\). Thus, the flux rope currents follow the twisted magnetic field. The angle of the magnetic field is \(B_\theta / B_z\) for the guide field dominated plasmas considered here. Since flux rope current produces \(B_\theta\), this means that the flux ropes rotate more for higher plasma currents and lower guide fields.

This rotation is exhibited for two flux ropes in Figure 6.4. At \(B_z = 1300\) G (a), are slightly rotated around the center of the experiment. When the guide field is lowered to \(B_z = 800\) G, the flux ropes exhibit a strong angle from the horizontal, indicating more rotational transform as expected. The distance between the two flux ropes is approximately constant as the guide field
Figure 6.4: The equilibrium of flux ropes with 500 A current calculated with Principle Component Analysis. (a) 1300 G guide field. Flux ropes are approximately force free ($J||B$), meaning they twist around each other as they travel down the experiment. (b) 800 G field. The flux ropes have expanded slightly and are more twisted around the axis of the experiment.

is lowered, indicating weak attractive forces between the two flux ropes. Currents have expanded slightly between the two guide field settings, creating slightly larger flux ropes at lower guide fields.

With three flux ropes (Figure 6.5), the story is a little different. At high guide field the flux ropes are independent structures. They are slightly rotated around the center of the experiment, as expected from the simple calculation. At lower guide field, however, the flux ropes have collided and formed a larger structure. The structure has three lobes, corresponding to the three flux ropes, which have conglomerated due to strong attractive forces. There is significant current between the flux rope.

To summarize, the flux rope equilibria exhibit a variety of structures depending on the rotational transform of the magnetic field, mutually attractive and repulsive forces between flux ropes, and the expansion of flux ropes. The structure of the flux rope equilibrium cannot be fully predicted from experimental controls, but certain scalings exist. Higher current, lower guide field flux ropes have a more helical nature. Plasmas with three flux ropes exhibit stronger attractive forces and merge more quickly than plasmas with two flux ropes. These equilibria influence fluctuations and
Figure 6.5: The equilibrium of flux ropes with 500 A current calculated with Principle Component Analysis. (a) 1300 G guide field. Flux ropes are approximately force free ($\mathbf{J} \parallel \mathbf{B}$), meaning they twist around each other as they travel down the experiment. (b) 800 G field. The flux ropes have expanded slightly and are more twisted around the axis of the experiment.

coccur, in turn, mediate the processes the form the equilibria.

### 6.3.2 Fluctuations

All flux rope plasmas exhibit significant fluctuations on top of the low frequency equilibrium described in the previous section. Observed flux rope behavior is complex and never develops coherent modes as seen with the hollow current equilibrium of Chapter 5. Instead, flux ropes appear to be turbulent, as seen in Figure 6.6. Plasmas with two flux ropes (Figure 6.6 (a)) and three flux ropes (Figure 6.6 (b)). Flux rope plasmas exhibit broadband, intermittent fluctuations at all parameters.

Figure 6.7 displays the rms fluctuation level, normalized by the current in each flux rope, measured by the external magnetic array for all two and three flux rope plasmas. Despite the incoherent fluctuations seen in Figure 6.6, plasmas at and currents have the same normalized fluctuation level dependent only on $B$. These fluctuations have no critical safety factor or plasma current.

The complexity and entropy scores (see Section 1.4.2) for every probe in every two and three flux rope plasma are shown in Figure 6.8. All signals lie in the chaotic portion of the plane, indicating chaotic interactions between flux ropes. Flux ropes with lower safety factors tend toward...
Figure 6.6: Spectograms of magnetic signals from 2 (a) and 3 (b) flux rope plasma with 500 A current per rope and a guide field of $B_z = 1000$ G. Spectograms are calculated with the Morlet wavelet. Both spectra exhibit broadband, intermittent fluctuations indicative of complex processes.

Figure 6.7: Normalized fluctuation levels of magnetic signals from 2 (a) and 3 (b) flux rope plasmas. Plasmas exhibit the same fluctuations characteristics at all values of $B$ despite wildly different plasma currents and safety factors.
Figure 6.8: The Complexity-Entropy measurements for all magnetic signals in the data sets, for 2 (a) and 3 (b) flux ropes. Each dot is a magnetic signal and signals are colored by the estimated safety factor of the rope. All flux rope plasmas are chaotic, landing in the upper central portion of the map, but low safety factor shots tend to have higher entropy.

higher entropy scores, indicating that stochastic processes are more common at higher plasma currents.

When searching for fluctuating eigenfunctions, Principle Component Analysis yields many complicated eigenfunctions with similar energy levels. This means that fluctuations aren’t simple structures that appear intermittently. Instead, they are independent complex structures.

Figures 6.6, 6.7, and 6.8 together show that flux ropes plasmas interact chaotically. This chaos is dynamically self-organized. The entropy of the chaos increases as \( q \) decreases.

To test the assumption of turbulence, we turn to wave number spectra calculated with fields measured by the internal magnetic array. Figure 6.9 presents wave number spectra for k-spectra for two plasmas with three flux ropes and 500 A of current per flux rope. The blue curve is at high guide field (high \( q \)) and the red curve is at low guide field (low \( q \)). Two features are important: The bump in the blue curve at (a) is the spectra signal of the individual flux ropes. The flux ropes are not distinguishable in the low field spectrum. The low field spectrum follows a power law \( E \propto k_{\perp}^{-2.5\pm0.5} \). This power law, combined with high experimental resistivity, indicates that fluctuations are chaotic and dynamic, but not fully developed turbulence.

Second, the largest scale modes (at low \( q \)) are significantly stronger in the low field spectrum.
Figure 6.9: Wave number spectra of magnetic signals plasmas with three flux ropes and 500 A current per flux rope. The blue spectrum is from a plasma with 1300 G guide field and the red spectrum is from a plasma with 500 G guide field. Two features are important: The bump in the blue curve at (a) is the spectra signal of the individual flux ropes. The flux ropes are not distinguishable in the low field spectrum. Second, the largest scale modes (at low q) are significantly stronger in the low field spectrum. The low field spectrum follows a power law $E \propto k_{\perp}^{-2.5}$.

This reduction of energy in small-scale modes and the increase of energy in large-scale modes is indicative of an inverse cascade. Traditionally associated with two-dimensional systems, the inverse cascade can also occur in three dimensional systems with strong guide field [25]. At this point it is unclear of the rise of large-scale modes is due to an inverse cascade or the presence of dissipation in the system.

### 6.4 Discussion

Plasmas were also analyzed visually with movies of plasma current evolution. Flux Rope bouncing and merging processes, observed in other experiments [66], were not seen. These processes may have been lost in the measured fluctuations or hidden by the limited spatial resolution on the internal probe array.

This Chapter presented measurements of the interactions of two and three line-tied flux ropes. Equilibria of flux rope plasmas have a three-dimensional helical structure that depends on the
rotational transform of the magnetic field. The flux ropes are relatively stable at high guide fields and safety factors, but dynamically merge as guide field is lowered.

Measured fluctuations in flux rope plasmas are intermittent, broadband and chaotic. At low safety factors, the fluctuations follow a steep spatial power law \( k^{-2.5} \) and exhibit characteristics of an inverse cascade. More measurements are required to distinguish the inverse cascade from other processes, such as diffusion.
Chapter 7

Conclusion

7.1 Summary

While currents are injected into the experiment at discrete locations, internal measurements presented in Chapter 3 suggest that screw pinch equilibria are approximately one dimensional for most of the experimental volume. Approximation holds for most of the experimental volume, but small deviations from 1D equilibria are inherent in the system. The question then arises: do approximately one-dimensional equilibria have approximately the same stability properties and truly one dimensional equilibria?

The answer to this question is a resounding no. Chapter 4 and Chapter 5 presented detailed measurements of self-organized states in complicated screw pinch equilibria. Chapter 4 presented the first laboratory zero-net-current equilibrium. The equilibrium undergoes a sub-critical transition to dynamic and chaotic fluctuations, possibly driven by fluctuations in the plasma source. The fluctuations use energy from the current drive to create impulsive magnetic reconnection events on a variety of scales. These reconnection events dynamically relax the plasma, creating a linearly stable three dimensional equilibrium.

In Chapter 5 a screw pinch with a hollow current profile was created. The hollow current equilibrium exhibits fluctuations at extremely high safety factors, possibly driven by resistive drift.
waves. As the magnetic field is lowered, the equilibrium bifurcates and relaxes into a helical mode that smooths the current profile. Relaxation occurs at a critical value of magnetic field, independent of plasma current. This state appears to be the first demonstration of a Taylor relax state [100] in a line-tied experiment. Measurements lack the required resolution to prove the conservation of helicity.

Finally, Chapter 6 compared the three dimensional equilibria of line-tied flux ropes to the approximately one dimensional equilibria in the previous chapters. Plasmas with two and three distinct flux ropes exhibit dynamic, turbulence-like interactions. The interactions appear to undergo a dynamic inverse cascade as flux ropes become unstable and merge. The role of other processes, such as diffusion, may be important for explaining these dynamics.

7.2 Future Work

7.2.1 Three Dimensional Measurements on LTRX

The internal magnetic probe array (Section 2.2.3) built for this dissertation allows the direct measurement of temporal and spatial dynamics at one axial location in the plasma. The high aspect ratio of the experiment allows the first order approximation $\partial / \partial z \to 0$. Clearly, this approximation is invalid globally. The external probe array, for example, reveals plasma oscillations with three dimensional structures (Figure 5.8).

Indeed, line-tying in the device means that instabilities, magnetic reconnection events, and turbulence are inherently three-dimensional processes. A full understanding of the processes studied in this dissertation requires three dimensional measurements of plasma behavior. For instance, flux rope interactions likely occur at several locations in $z$ and the equilibrium in the hollow screw pinch necessarily varies in $z$. The current sheet measured in the zero-net current plasma must have three dimensional structure, modifying the reconnection rate.

Additional probe arrays are necessary to measure these effects in the plasma. The simplest way to implement three dimensional measurements would be the installation of two additional box
ports at \( z = 52 \) cm and \( z = 156 \) cm. A copy of the internal probe array at each location would provide coarse three dimensional measurements of structures in the plasma. Since experimental \( B_z/B_\perp > 10 \), axial resolution of 52 cm could just capture the dynamics of fluctuations with radial size of 5 cm.

The addition of electrostatic arrays measuring ion saturation current and floating potential would be an excellent addition to the diagnostic capabilities of the device. Co-locating these probes with the magnetic probes would allow for the direct correlation of magnetic, density, and electric field fluctuations. This would allow more direct measurements into the statistics of fluctuations and the physics of magnetic reconnection in the device.

In total, such a system would require 576 additional magnetic probes and 576 electrostatic probes. To measure these signals would require an additional 1152 amplification circuits and digitization channels. For reference, a set of D-Tacq brand ACQ-196 cards and crates to measure those signals would cost approximately $150,000 dollars on top of probe and amplifier costs. While the project is likely too expensive for construction in the next few years, it is likely the best way to understand the fully three-dimensional dynamics of instability and turbulence in this experiment.

### 7.2.2 Parker Instability

One experiment was conceived during the course of this dissertation but never performed. The Parker instability [103] is an interchange-like instability where the magnetic field in a galactic disk becomes buoyantly unstable and drags plasma above the disk. Khalzov et al. [104] recently demonstrated that fast azimuthal rotation (\( M > 5 \)) in a screw pinch can destabilize the Parker instability with the centripetal force substituting for gravity. Such rotating speeds should be attainable in the device by biasing the center gun and anode by a few kilovolts relative to the outer guns. Such a system would require better electrical isolation in the bias system, but should be feasible.
7.2.3 Nanoflare Heating

The measurements presented in Chapters 4 and 6 indicate turbulence-like fluctuations and magnetic reconnection on a variety of scales. These reconnection events likely heat the plasma, but confinement on the device is too poor to measure the effect. A better confined system would be required to compare data to theories of coronal heating [43, 44].

An array of washer guns could be installed on the Madison Plasma Dynamo Experiment (MPDX) [105] vacuum vessel. The interactions of flux ropes could then be studied with advanced optical diagnostics to search for nanoflare heating of the plasma. The excellent confinement time in MPDX would allow precise measurements of heating.

7.3 Conclusion

This dissertation presented studies of stability and chaotic behavior in the line-tied screw pinch with several complex equilibria. These equilibria are related to astrophysical and fusion plasmas, and exhibit many complex, dynamic phenomena. The flexibility and modularity of the Line-tied Reconnection Experiment present a unique opportunity for studying these phenomena in a controlled laboratory setting. Future work may elucidate the three-dimensional nature of such basic physics phenomena, but would require significant monetary investment in improved diagnostics.

Thank you for reading. I hope you’ve learned something from my dissertation.
Appendices
Appendix A

Spoiled Integrators

This appendix describes the theory and function of the custom spoiled integrators built for the Internal Probe Array. The theory is adapted from a private communication from John Sarff.

A.1 Theory of the Spoiled Integrator

Active analog integration using an operational amplifier (op-amp) is a useful tool for interpreting the signal from a magnetic pickup loop. It allows direct measurement of the local magnetic field while avoiding the low frequency errors associated with numerical integration. The primary challenge for the op-amp based integrator is the accumulation of charge on the integrating capacitor due to finite input offset voltage on the op-amp. Unchecked, the output voltage of the circuit will “drift” to the power supply voltage and saturate, rendering the circuit useless for measurement.

The simplest way to manage this drift is to add a resistor across the feedback capacitor, as shown in Figure A.1. The input offset voltage of the op-amp is modeled as a DC voltage source. When the input voltage $V_{in} = 0$, $V_{out} = (R_2/R_1 + 1)V_{io}$. As long as $V_{out} < \approx 1$ Volt, $R_2$ sucessfully mitigates the circuit “drift” while allowing ample power supply headroom for a magnetic pulse.

Unfortunately, the addition of $R_2$ means that the feedback loop of the circuit acts as a low pass filter with $\tau = R_2C$. Pulses longer than $\tau$ are filtered out by the circuit. Thus, the spoiled integrator
is only useful for shorter pulse experiments, where moderate values of \( R_2 \) and \( C \) can produce a \( \tau \) comparable to the experimental pulse.

For intermediate values of \( \tau \) (i.e., \( \tau \approx t_p \)), the effect of the low pass filter can be removed in post processing. To separate the contributions of offset and input voltage, we assume that \( V_{in} \) has no time averaged component, while \( V_{io} \) has no fluctuating component. Then, \( V_{out} = \tilde{V}_{out} + \bar{V}_{out} \) where \( \tilde{V}_{out} \) is the fluctuating component and \( \bar{V}_{out} \) is the time-averaged component. The time averaged component is:

\[
\bar{V}_{out} = \left( \frac{R_2}{R_1} + 1 \right) V_{io} \tag{A.1}
\]

While the fluctuating component is:

\[
\tilde{V}_{out} = -\frac{1}{R_1 C} \int V_{in} - \frac{1}{R_2 C} \int \tilde{V}_{out} \tag{A.2}
\]

The first term on the right hand side of Eqn. A.2 is proportional to the magnetic field at the coil location. Thus,

\[
B \propto \tilde{V}_{out} + \frac{1}{R_2 C} \int \tilde{V}_{out} \tag{A.3}
\]

To correct the circuit output to find \( B \), one simply removes a pre-shot average from the signal to
Figure A.2: Traces from a prototype spoiled integrator, showing magnetic data from a plasma discharge.

Next, the numerical integral of $\tilde{V}_{out}$ is added to the signal, via Eqn. A.3. This yields a signal proportional to $B$ based on the area of the probes and the gain of the circuit. The process is illustrated in Figure A.2.

### A.2 Circuit Design Used on LTRX

A set of custom spoiled integrator circuits was built to integrate the Internal Array. They were designed using the open source KiCAD program. Custom, surface mount printed circuit boards were made for easy circuit assembly. Spare boards are available in the PCX/LTRX control room and the design files are available on the blackout server under /designs/integrators.
Appendix B

Fitting Magnetic Data

Because the probe Array described in 2.2.3 is not perfectly calibrated or aligned, a divergence cleaning routine was developed to fit the data and calculate $J_z$.

B.1 Fitting Model

First, assume a cylindrical plasma with $L >> a$. To first order we will assume that derivatives with respect to $z$ are negligible. We can then express the in-plane components of magnetic field in terms of the vector potential:

$$\vec{B} = \nabla \times \vec{A}$$  \hspace{1cm} (B.1)

$$B_r(r, \theta) = \frac{1}{r} \frac{\partial A_z}{\partial \theta}$$ \hspace{1cm} (B.2)

$$B_\theta(r, \theta) = -\frac{\partial A_z}{\partial r}$$ \hspace{1cm} (B.3)
and the current as

$$\mu_0 \vec{J} = \nabla \times (\nabla \times \vec{A})$$  \hspace{1cm} (B.4)$$

$$= \nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$$  \hspace{1cm} (B.5)$$

$$\mu_0 J_z (r, \theta) = -\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial A_z}{\partial r} \right) - \frac{1}{r^2} \frac{\partial^2 A_z}{\partial \theta^2} - \frac{\partial^2 A_z}{\partial r^2} - \frac{1}{r} \frac{\partial A_z}{\partial r} - \frac{1}{r^2} \frac{\partial^2 A_z}{\partial \theta^2}$$  \hspace{1cm} (B.6)$$

To reduce the error from misalignment and miscalibration we will fit

$$A_z (r, \theta) = F_0 (r) + \sum_{m=1}^{\infty} \left[ F_m (r) \cos (m \theta) + G_m (r) \sin (m \theta) \right]$$  \hspace{1cm} (B.7)$$

Where $F_m (r)$ and $G_m (r)$ are yet to be determined functions. We can then express

$$B_r (r, \theta) = \sum_{m=1}^{\infty} \left[ -\frac{m F_m (r)}{r} \sin (m \theta) + \frac{m G_m (r)}{r} \cos (m \theta) \right]$$  \hspace{1cm} (B.8)$$

$$B_\theta (r, \theta) = -\frac{\partial F_0 (r)}{\partial r} - \sum_{m=1}^{\infty} \left[ \frac{\partial F_m (r)}{\partial r} \cos (m \theta) + \frac{\partial G_m (r)}{\partial r} \sin (m \theta) \right]$$  \hspace{1cm} (B.9)$$

and in Cartesian coordinates

$$B_x (r, \theta) = B_r (r, \theta) \cos \theta - B_\theta (r, \theta) \sin \theta$$  \hspace{1cm} (B.10)$$

$$= \sum_{m=1}^{\infty} \left[ -\frac{m F_m (r)}{r} \sin (m \theta) \cos \theta + \frac{m G_m (r)}{r} \cos (m \theta) \cos \theta \right] + \frac{\partial F_0 (r)}{\partial r} \sin \theta + \sum_{m=1}^{\infty} \left[ \frac{\partial F_m (r)}{\partial r} \cos (m \theta) \sin \theta + \frac{\partial G_m (r)}{\partial r} \sin (m \theta) \sin \theta \right]$$  \hspace{1cm} (B.11)$$
\[ B_z(r, \theta) = B_r(r, \theta) \sin \theta + B_\theta(r, \theta) \cos \theta \] \hspace{1cm} (B.12)

\[
\begin{align*}
&= \sum_{m=1}^{\infty} \left[ \frac{-mF_m(r)}{r} \sin(m\theta) \sin \theta + \frac{mG_m(r)}{r} \cos(m\theta) \sin \theta \right] \\
&- \frac{\partial F_0(r)}{\partial r} \cos \theta - \sum_{m=1}^{\infty} \left[ \frac{\partial F_m(r)}{\partial r} \cos(m\theta) \sin \theta + \frac{\partial G_m(r)}{\partial r} \sin(m\theta) \cos \theta \right] \\
&\hspace{1cm} (B.13)
\end{align*}
\]

Regularity on axis requires that \( J_z \) be single valued and finite at \( r=0 \), i.e.

\[
0 = \lim_{r \to 0} \frac{\partial}{\partial \theta} J_z(r, \theta) \hspace{1cm} (B.14)
\]

\[
= \lim_{r \to 0} \left[ \sum_{m=1}^{\infty} \left( \left( \frac{\partial^2 F_m(r)}{\partial r^2} + \frac{1}{r} \frac{\partial F_m(r)}{\partial r} - \frac{m^2}{r^2} F_m(r) \right) m \sin(m\theta) \right) \\
- \left( \frac{\partial^2 G_m(r)}{\partial r^2} + \frac{1}{r} \frac{\partial G_m(r)}{\partial r} - \frac{m^2}{r^2} G_m(r) \right) m \cos(m\theta) \right) \right] \hspace{1cm} (B.15)
\]

\[
-\infty < J_z(r, \theta) < \infty \hspace{1cm} (B.16)
\]

\[
-\infty < \lim_{r \to 0} \left[ \sum_{m=1}^{\infty} \left( \left( \frac{\partial^2 F_0(r)}{\partial r^2} + \frac{1}{r} \frac{\partial F_0(r)}{\partial r} \right) + \left( \frac{\partial^2 F_m(r)}{\partial r^2} + \frac{1}{r} \frac{\partial F_m(r)}{\partial r} - \frac{m^2}{r^2} F_m(r) \right) \cos(m\theta) \right) \\
- \left( \frac{\partial^2 G_m(r)}{\partial r^2} + \frac{1}{r} \frac{\partial G_m(r)}{\partial r} - \frac{m^2}{r^2} G_m(r) \right) \sin(m\theta) \right] < \infty \hspace{1cm} (B.17)
\]

This means:

\[
\lim_{r \to 0} \left[ \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{m^2}{r^2} \right] \{F_m, G_m\} = 0 \hspace{1cm} (B.18)
\]

\[
\lim_{r \to 0} \left[ \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} \right] F_0 < \infty \hspace{1cm} (B.19)
\]

The insulating vacuum vessel requires that \( \lim_{r \to R} J_z = 0 \) which gives the outer boundary
conditions
\[
\lim_{r \to R} \left[ \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{m^2}{r^2} \right] \{F_m, G_m\} = 0
\] (B.20)

At this point we will introduce the assumption that \( F \) and \( G \) are \( n \)-order polynomial functions, \( \text{i.e.} \{F_m, G_m\} = \sum_{i=0}^{n} a_i x^i \). Eqns. B.18 and B.19 then become

\[
\lim_{x \to 0} \left[ a_1 x + \sum_{i=2}^{n} a_i x^i \right] < \infty \] (B.21)

\[
\lim_{x \to 0} \left[ a_1 x - \frac{m^2 a_1}{x} - \frac{m^2 a_0}{x^2} + \sum_{i=2}^{n} a_i (r^2 - m^2) x^i \right] = 0 \] (B.22)

These equations, along with Eqn. B.20 provide a set of constraints on \( f_m \in F_m, G_m \) which are summarized in Table B.1

<table>
<thead>
<tr>
<th>( m )</th>
<th>( f_m(0) )</th>
<th>( f'_m(0) )</th>
<th>( f''_m(0) )</th>
<th>( f'_m(R) )</th>
<th>( f''_m(R) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>NC</td>
<td>0</td>
<td>NC</td>
<td>NC</td>
<td>(-f'_0(R)/R)</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>NC</td>
<td>0</td>
<td>NC</td>
<td>(f'_1(R)/R^2 - f'_1(R)/R)</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>NC</td>
<td>NC</td>
<td>(4f'_2(R)/R^2 - f'_2(R)/R)</td>
</tr>
<tr>
<td>( m &gt; 2 )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>NC</td>
<td>(m^2 f'_m(R)/R^2 - f'_m(R)/R)</td>
</tr>
</tbody>
</table>

Table B.1: A summary on the boundary conditions of the fitting functions. ‘NC’ indications that the boundary isn’t constrained.

Now that the appropriate boundary conditions have been defined, the functions \( F_m \) and \( G_m \) need to be defined. To accomplish this we will expand \( F_m \) and \( G_m \) in terms of cubic splines whose outer boundary conditions match Eqn. B.20 (these cubic splines are described in detail in Sec. B.2).

\[
F_m(r) = \delta(m)c_{m0}f_{m0}(r) + \sum_{n=1}^{N} c_{mn} f_{mn}(r) \] (B.23)

\[
G_m(r) = \sum_{n=1}^{N} d_{mn} f_{mn}(r) \] (B.24)

Where the \( f_{mn} \) are the splines pass through the points \([r_0, r_1, ... r_N]\) and the vector \( \hat{y}_n \) (where \( y[i \neq n] = 0 \) and \( y[n] = 1 \)) and match the boundary conditions given in Table B.1

Using this set of basis functions for \( F \) and \( G \) allows us to match the boundary conditions automatically and cast the fitting as a linear problem. With this expansion Eqn. B.7 can be rewritten.
as

\[ A_z(r, \theta) = \sum_{n=0}^{N} c_{0n} f_{0n}(r) \]

\[ + \sum_{m=1}^{\infty} \left[ \left( \sum_{n=1}^{N} c_{mn} f_{mn}(r) \right) \cos(m\theta) \right. \]

\[ \left. - \left( \sum_{n=1}^{N} d_{mn} f_{mn}(r) \right) \sin(m\theta) \right] \]

(B.25)

And the B’s become:

\[ B_r(r, \theta) = \sum_{m=1}^{\infty} \left[ \left( \sum_{n=1}^{N} -c_{mn} \frac{m f_{mn}(r)}{r} \right) \sin(m\theta) \right. \]

\[ + \left. \left( \sum_{n=1}^{N} d_{mn} \frac{m f_{mn}(r)}{r} \right) \cos(m\theta) \right] \]

(B.26)

\[ B_\theta(r, \theta) = -\sum_{n=0}^{N} c_{0n} f'_{0n}(r) \]

\[ + \sum_{m=1}^{\infty} \left[ \left( \sum_{n=1}^{N} -c_{mn} f'_{mn}(r) \right) \cos(m\theta) \right. \]

\[ + \left. \left( \sum_{n=1}^{N} -d_{mn} f'_{mn}(r) \right) \sin(m\theta) \right] \]

(B.27)

while \( J_z \) is

\[ \mu_0 J_z(r, \theta) = \sum_{n=0}^{N} c_{0n} \left( -f''_{0n}(r) - \frac{1}{r} f'_{0n}(r) \right) \]

\[ + \sum_{m=1}^{\infty} \left[ \sum_{n=1}^{N} c_{mn} \left( -f''_{mn}(r) - \frac{1}{r} f'_{mn}(r) + \frac{m^2}{r^2} f_{mn}(r) \right) \cos(m\theta) \right. \]

\[ + \left. \left( \sum_{n=1}^{N} d_{mn} \left( -f''_{mn}(r) - \frac{1}{r} f'_{mn}(r) + \frac{m^2}{r^2} f_{mn}(r) \right) \sin(m\theta) \right) \right] \]

(B.28)

Because the Magnetic Array is aligned with respect to \( \hat{x}, \hat{y}, \hat{z} \) instead of \( \hat{r}, \hat{\theta}, \hat{z} \) we need to perform a coordinate transformation to utilize this model. Rotating the vectors,
\[
B_x(r, \theta) = \left( \sum_{n=0}^{N} c_{0n} f_{0n}'(r) \right) \sin \theta
+ \sum_{m=1}^{\infty} \left[ \sum_{n=1}^{N} c_{mn} \left( -\frac{m f_{mn}(r)}{r} \sin m \theta \cos \theta + f_{mn}'(r) \cos m \theta \sin \theta \right) \right. \\
+ \left. \sum_{n=1}^{N} d_{mn} \left( \frac{m f_{mn}(r)}{r} \cos m \theta \cos \theta + f_{mn}'(r) \sin m \theta \sin \theta \right) \right] \tag{B.29}
\]

\[
B_y(r, \theta) = \left( -\sum_{n=0}^{N} c_{0n} f_{0n}'(r) \right) \cos \theta \\
+ \sum_{m=1}^{\infty} \left[ \sum_{n=1}^{N} c_{mn} \left( -\frac{m f_{mn}(r)}{r} \sin m \theta \sin \theta - f_{mn}'(r) \cos m \theta \cos \theta \right) \right. \\
+ \left. \sum_{n=1}^{N} d_{mn} \left( \frac{m f_{mn}(r)}{r} \cos m \theta \sin \theta - f_{mn}'(r) \sin m \theta \cos \theta \right) \right] \tag{B.30}
\]

Equations B.29 and B.30 can be used to define a matrix equation

\[
A \cdot c = B \tag{B.31}
\]

Where \( c \) is the vector described by \( [\tilde{c}, \hat{c}_0, c_{01}, c_{02}, \ldots, c_{M,N-2}, d, \ldots] \), \( B \) is the magnetic field measured by the probe array with each element in the vector representing a probe with a certain position and direction, and \( A \) is the matrix defined by Eqns. B.29 and B.30 and the measurement positions and directions of the \( B \). To find the values of \( c \), a pseudo-inverse is performed on \( A \) using the SVD and the resulting matrix \( \tilde{A}^{-1} \) is operated on \( B \).
**B.2 Cubic Splines**

Let \((x_i, y_i)\) where \(i = 0, 1, \ldots, n\) be \(n + 1\) points and

\[
q_i = (1-t) y_{i-1} + t y_i + t (1-t) (a_i (1-t) + b_i t)
\]  

(B.32)

be \(n\) third degree polynomials interpolating \(y\) in the interval \(x_{i-1} \leq x \leq x_i\) such that \(q_i(x_i) = q_{i-1}(x_i)\)

then the \(n\) polynomials together define a differentiable function in the interval \(x_0 \leq x \leq x_n\) where

\[
t = \frac{x - x_{i-1}}{x_i - x_{i-1}}
\]  

(B.33)

\[
a_i = k_{i-1}(x_i - x_{i-1}) - (y_i - y_{i-1})
\]  

(B.34)

\[
b_i = -k_i(x_i - x_{i-1}) + (y_i - y_{i-1})
\]  

(B.35)

for \(i = 0, 1, \ldots, n\) and

\[
k_0 = q'_1(x_0)
\]  

(B.36)

\[
k_i = q'_i(x_i) = q'_{i+1}(x_i) \quad i = 1, \ldots, n-1
\]  

(B.37)

\[
k_n = q'_n(x_n)
\]  

(B.38)

If the sequence \(k_0, k_1, \ldots, k_n\) is such that in addition \(q''_i(x_i) = q''_{i-1}(x_i)\) for \(i = 0, 1, \ldots, n\) the resulting function will have a continuous second derivative. It follows that this is the case if and only if

\[
\frac{k_{i-1}}{x_i - x_{i-1}} + \left(\frac{1}{x_i - x_{i-1}} + \frac{1}{x_{i+1} - x_i}\right) 2k_i + \frac{k_{i+1}}{x_{i+1} - x_i} = 3 \left(\frac{y_i - y_{i-1}}{(x_i - x_{i-1})^2} + \frac{y_{i+1} - y_i}{(x_{i+1} - x_i)^2}\right)
\]  

(B.39)

Which defines \(n - 1\) equations for the interior slopes of the polynomials.
Natural Cubic splines are defined such that \( q'_0(x_0) = q''_{n-1}(x_n) = 0 \). This gives

\[
q''_0(x_0) = 2 \frac{3(y_1 - y_0) - (k_1 + 2k_0)(x_1 - x_0)}{(x_1 - x_0)^2} = 0
\]

\[
q''_n(x_n) = -2 \frac{3(y_n - y_{n-1}) - (2k_n + k_{n-1})(x_n - x_{n-1})}{(x_n - x_{n-1})^2} = 0
\]

\[
\frac{2}{x_1 - x_0} k_0 + \frac{1}{x_1 - x_0} k_1 = 3 \frac{y_1 - y_0}{(x_1 - x_0)^2}
\]

\[
\frac{1}{x_n - x_{n-1}} k_{n-1} + \frac{2}{x_n - x_{n-1}} k_n = 3 \frac{y_n - y_{n-1}}{(x_n - x_{n-1})^2}
\]

Which, along with Eqn. B.39 gives \( n+1 \) linear equations for \( n+1 \) unknowns.

Second, we can ‘clamp’ the spline by specifying the boundary derivatives \( k_0 \) and \( k_n \), reducing the system to \( n-1 \) equations, solvable by Eqn. B.39.

To match the boundary conditions in Section B.1, we can define more complicated spline boundaries. First, for \( m = 0 \) and \( m = 2 \) we require a set \( k_0 = 0 \) and the condition

\[
\lim_{x \to x_n} \left[ \frac{1}{x} \frac{\partial}{\partial x} \left( x \frac{\partial q}{\partial x} \right) - \frac{m^2}{x^2} q \right] = 0
\]

Inserting \( q_n \) defined by B.32 this becomes

\[
0 = q''_n(x_n) + \frac{1}{x_n} q'_n(x_n) - \frac{m^2}{x_n^2} q_n(x_n)
\]

\[
= -2 \frac{3(y_n - y_{n-1}) - (2k_n + k_{n-1})(x_n - x_{n-1})}{(x_n - x_{n-1})^2}
\]

\[
+ \frac{k_n}{x_n} - \frac{m^2}{x_n^2} y_n
\]

which yields

\[
\left( \frac{4}{x_n - x_{n-1}} + \frac{1}{x_n} \right) k_n + \frac{2}{x_n - x_{n-1}} k_{n-1} = \frac{6(y_n - y_{n-1})}{(x_n - x_{n-1})^2} + \frac{m^2 y_n}{x_n^2}
\]

Eqns. B.39 and B.47 form a set of \( n \) equations that, together with a set \( k_0 = 0 \), fully define the system of splines.
The regularity conditions on $m = 1$ require that the splines satisfy B.44 and

$$\lim_{x \to x_0} \left[ \frac{1}{x} \frac{\partial}{\partial x} \left( x \frac{\partial q}{\partial x} \right) - \frac{1}{x^2} q \right] = 0 \quad (B.48)$$

For the special case where $y_0 = x_0 = 0$, this reduces to

$$2k_0 + k_1 = 3\frac{y_1}{x_1} \quad (B.49)$$

Eqns. B.39, B.47, and B.49 form a set of n equations that fully define the system of splines.

### B.3 Quartic Splines

The regularity boundary conditions shown in Table B.1 over define the system of cubic splines described in the previous section for $m > 2$. Additionally, a piecewise cubic fit to $A_z$ results in a likewise linear fit to $J_z$ which is not ideal for fitting current sheets. To overcome these problems this section develops the theory of quartic (4th order) splines.

Let $(x_i, y_i)$ where $i = 0, 1, ..., n$ be $n + 1$ points and $q_i(x)$ $i = 1, 2, ..., n$ be $n$ fourth degree polynomials interpolating $y$ on the interval $x_{i-1} \leq x \leq x_i$ such that

- $q_i(x_i) = q_{i+1}(x_i) = y_i, \quad i \in 0, ..., n \quad (B.50)$
- $q'_i(x_i) = q'_{i+1}(x_i) = k_i, \quad i \in 1, ..., n - 1 \quad (B.51)$
- $q''_i(x_i) = q''_{i+1}(x_i) = l_i, \quad i \in 1, ..., n - 1 \quad (B.52)$
- $q'''_i(x_i) = q'''_{i+1}(x_i) = z_i, \quad i \in 1, ..., n - 1 \quad (B.53)$

These equations provide $5n - 3$ constraints on $5n$ equations, 3 boundary conditions will need to be set. To simplify the system of equations we can write:

$$q'''_i(x) = \frac{z_i}{x_i - x_{i-1}} (x - x_{i-1}) + \frac{z_{i-1}}{x_i - x_{i-1}} (x_i - x) \quad (B.54)$$
For simplicity, we will assume \( h = h_i = x_i - x_{i-1} \) for all \( i \). Integrating and carefully choosing integration constants we get

\[
q_i(x) = \frac{z_i}{24h}(x - x_{i-1})^4 - \frac{z_{i-1}}{24h}(x_i - x)^4 \\
+ \left( -\frac{z_i}{24h} + \frac{y_i}{h} \right) (x - x_{i-1}) \\
+ \left( \frac{z_{i-1}}{24h} + \frac{y_{i-1}}{h} \right) (x_i - x) \\
+ C_i(x - x_{i-1})(x_i - x)
\]  

(B.55)

which satisfies the boundary conditions on the function and its third derivative. For simplicity, we will assume \( h = h_i = x_{i+1} - x_i \) for all \( i \) (this is a valid simplification for the fitting we are doing).

To satisfy the boundary conditions on the first and second derivatives, we differentiate:

\[
q'_i(x) = \frac{z_i}{6h}(x - x_{i-1})^3 + \frac{z_{i-1}}{6h}(x_i - x)^3 \\
+ \left( -\frac{z_i}{24h} + \frac{y_i}{h} \right) - \left( \frac{z_{i-1}}{24h} + \frac{y_{i-1}}{h} \right) \\
+ C_i(x_i + x_{i-1} - 2x)
\]

\[
q''_i(x) = \frac{z_i}{2h}(x - x_{i-1})^2 - \frac{z_{i-1}}{2h}(x_i - x)^2 - 2C_i
\]

Next, letting \( q''_i(x_i) = q''_{i+1}(x_i) \) and

\[
\frac{z_i}{2}h - 2C_i = -\frac{z_i}{2}h - 2C_{i+1} \\
\rightarrow C_{i+1} = C_i - \frac{z_i}{2}h
\]  

(B.56)

Eqn. B.56 defines a recursive equation for \( C_i \). We can re-write this:

\[
C_i = C_1 - \sum_{j=1}^{i-1} \frac{z_j}{2h}
\]  

(B.57)
Next, setting $q_i'(x_i) = q_{i+1}'(x_i)$

\[
\frac{z_i h^2}{8} - \frac{z_{i-1} h^2}{24} + \frac{y_i - y_{i-1}}{h} - h \left( C_1 - \sum_{j=1}^{i-1} \frac{z_j}{2} h \right) = \frac{z_i h^2}{8} - \frac{z_{i+1} h^2}{24} + \frac{y_{i+1} - y_i}{h} + h \left( C_1 - \sum_{j=1}^{i} \frac{z_j}{2} h \right)
\]

\[
\rightarrow (\frac{z_{i+1}}{24} + \frac{z_i}{2} - \frac{z_{i-1}}{24}) h^2 + h^2 \sum_{j=1}^{i} z_j - 2C_1 h = \frac{y_{i+1} - 2y_i + y_{i-1}}{h}
\]

(B.58)

Next, we need to choose three boundary conditions to finish the system of equations. First, we match the boundary conditions as $x \rightarrow x_n$. From Table B.1 and B.55:

\[
q''_n(x_n) = \frac{m^2 q_n(x_n)}{x_n^2} - \frac{q'_n(x_n)}{x_n}
\]

\[
\frac{z_n h^2}{2} - 2C_n = \frac{m^2 y_n}{x_n^2} - \left( \frac{z_n h^2}{8x_n} - \frac{z_{n-1} h^2}{24x_n} + \frac{y_n - y_{n-1}}{hx_n} - \frac{C_n h}{x_n} \right)
\]

\[
\rightarrow \left( \frac{h}{2} + \frac{h^2}{8x_n} \right) z_n = \frac{h^2}{24x_n} z_{n-1} - \left( 2 + \frac{h}{x_n} \right) \left( C_1 - \sum_{j=1}^{n-1} \frac{z_j}{2} h \right) = \frac{m^2 y_n}{x_n^2} - \frac{y_n - y_{n-1}}{hx_n}
\]

(B.59)

At $x_0$ we have 3 different boundary conditions dependent on $m$. For $m = 0, 2$, we set $q''_1(x_0) = q'_1(x_0) = 0$

\[
z_0 = 0 \quad \text{(B.60)}
\]

\[
q'_1(x_0) = 0 = \frac{z_1 h^2}{24} + \frac{y_1 - y_0}{h} + C_1 h
\]

\[
\rightarrow C_1 = \frac{z_1 h}{24} - \frac{y_1 - y_0}{h^2}
\]

(B.61)

For $m = 1$, $q'''_1(x_0) = q''_0(x_1) = 0$

\[
z_0 = 0 \quad \text{(B.62)}
\]

\[
q''_1(x_0) = 0 = -2C_1
\]

\[
\rightarrow C_1 = 0 \quad \text{(B.63)}
\]
Finally, for $m > 2$, $q'_0(x_0) = q''_0(x_0) = 0$

\[ q'_1(x_0) = 0 = \frac{z_0 h^2}{8} - \frac{z_1 h^2}{24} + \frac{y_1 - y_0}{h} + C_1 h \]

\[ q''_1(x_0) = 0 = -\frac{z_0 h}{2} - 2C_1 \]

\[ \rightarrow C_1 = -\frac{z_0 h}{4} \]  

\[ \frac{z_0 h^2}{8} + \frac{z_1 h^2}{24} = \frac{y_1 - y_0}{h} \]  

(B.64)

(B.65)

Together, Eqns. B.59, B.60, B.61, B.62, B.63, B.64, and B.64 provide 3 constraints on the equations for every $m$, fully constraining the quartic spline problem.
Appendix C

Czerny Turner Spectrometer

A description of a 1.5m Czerny-Turner Monochrometer from the 1960’s is given. This spectrometer was given to UW-Madison from a Swedish group who no longer used the device. Optical components are described in detail, the theory of the achievable resolution is given, and expected results from certain uses are proposed. The group recently purchased an Intensified Charged Coupled Device Camera from Andor which may alleviate the sensitivity and resolution limitations described below.

The spectrometer end of the fiber is collimated and focused onto the slit with an achromatic doublet lens. The entrance slit is mounted to the exterior of the spectrometer and is easily changed. Slits of width 10, 20, 50, and 100 μm are available. Two spherical mirrors of \( f = 1.5m \) are used in the spectrometer. One 4 inch tall by 4 inch wide mirror (M1 in Fig. C.1) collimates the light from the entrance slit while one 4 inch tall by 20 inch wide mirror (M2 in Fig. C.1) focuses the dispersed light onto the detector. The Diffraction grating used in the experiment was constructed by Bausch & Lomb and certified on Jun 12, 1974. It has 1200 grooves/mm on a flat, rectangular ruled area of 102mm × 102mm and a blaze angle of 17°27′ at a wavelength of 5000Å. The grating resolution is of 90% the theoretical. The theoretical resolving power for this spectrometer is \( R = \frac{\lambda}{\delta \lambda} = nN = 122400 \) (the total number of lines), thus the actual resolving power is \( R \approx 110160 \) or \( \delta \lambda \approx 0.045\text{Å} \) at \( \lambda = 5000\text{Å} \). The detector currently used on the spectrometer is a CCD-2000M from ALPHALAS GmbH. This is a 1-D linear CCD with 2048 pixels of 14 μm width and 200 μm height. Integration time and data readout are controlled by a National Instruments PXi create
running Windows XP and Labview. The CCD requires a white-light back fill for linear pixel response. We provide this back fill with an adjustable incandescent lamp in front of the diffraction grating.

Because the collimation and focusing mirrors are spherical the spectrometer is prone to certain aberrations. The first important aberration in the Czerny-Turner design is coma. Coma produces an image distortion proportional to $\tan \theta$ where $\theta$ is the angle between the incoming light and the optical axis of the mirror/lens in question. If the beams are set up as shown in Fig. C.1, $\theta_{M1} = -\theta_{M2}$ and the distortions of coma disappear. If, instead, the detector is placed at larger $y$, coma distorts the lines. Thus line shape measurements need to be taken such that the detector and entrance slit are equidistant from the optical axis while line intensity measurements can be taken with the detector at larger $y$.

The other important source of error in the Czerny-Turner design is Astigmatism. The most important result of astigmatism for the Czerny-Turner is a curved focal plane. From Sec 4.5 of [106],

$$R_{CT} = \left( \frac{2}{f} + \frac{3Z}{f} + \frac{3Z^2}{4f^3} \right)^{-1}$$  \hspace{1cm} (C.1)$$

where $R_{CT}$ is the radius of curvature of the tangential focal plane (where spectra are focused) and
$Z = -0.75m$ is the location of the diffraction grating relative to the mirrors. This gives $R_{Cr} = 2.182m$. If a detector is placed such that the entrance slit and detector are equidistant from the optical axis (as shown in Fig. C.1), then the detector will need to sit 3.3 mm in front of the entrance slit at an angle of 3.1° such that the spectra are focused on the detector.

![Figure C.2: Schematic of the Czerny-Turner Diffraction Grating.](image)

Figure C.2 shows the relevant directions and angles required to calculate the achievable resolution of the spectrometer. Because the collimating mirror has a radius of curvature of 3 m centered at a point along the optical axis 1.5 m behind the entrance slit, it is straightforward to calculate $\beta = \frac{v}{f} = 0.08$ for the beam which strikes the center of the diffraction grating. For the wavelength aligned,

$$\beta' = \beta$$  \hspace{1cm} (C.2)

The grating equation states

$$\frac{n\lambda}{a} = \sin \theta + \sin \phi$$  \hspace{1cm} (C.3)

$$\phi = \sin^{-1} \left( \frac{n\lambda}{a} - \sin \theta \right)$$  \hspace{1cm} (C.4)

where $a = 833nm$ is the spacing between rulings on the diffraction grating, $n = 1$ is the diffraction order, $\theta = \beta + \alpha$ is the angle of the incoming beam relative to normal, and $\phi$ is the angle of the outgoing beam relative to normal. $\theta$ and $\phi$ are both positive when on opposite sides of the normal as shown in Fig. C.2. The angular dispersion of a wavelength at $\lambda = \lambda_0 + \delta \lambda$ can be found by
differentiating (C.4) with respect to wavelength while $\theta = constant$. This gives

$$\frac{d\phi}{d\lambda} = -\frac{n}{a}\left(1 - \left(\frac{n\lambda}{a} - \sin \theta\right)^2\right)^{1/2}$$  \hspace{1cm} (C.5)

$$\frac{d\phi}{d\lambda} = -\frac{n}{a}\sec \phi$$  \hspace{1cm} (C.6)

and the linear dispersion along the focal plane will be

$$dl = -fd\phi = d\lambda \frac{n f}{a} \sec \phi$$  \hspace{1cm} (C.7)

Using the geometrical constraints

$$\theta = \alpha + \beta$$  \hspace{1cm} (C.8)

$$\phi = \alpha - \beta'$$  \hspace{1cm} (C.9)

and (C.2) we can rewrite (C.3) and (C.4) as

$$\frac{n\lambda_0}{a} = \sin (\alpha + \beta) + \sin (\alpha - \beta)$$  \hspace{1cm} (C.10)

$$\frac{n\lambda_0}{a} = \sin \alpha \cos \beta + \cos \alpha \sin \beta + \sin \alpha \cos \beta \cos \alpha \sin \beta$$  \hspace{1cm} (C.11)

$$\frac{n\lambda_0}{a} = 2 \cos \beta \sin \alpha$$  \hspace{1cm} (C.12)

$$\alpha = \sin^{-1}\left(\frac{n\lambda_0}{2a \cos \beta}\right)$$  \hspace{1cm} (C.13)

for some aligned $\lambda_0$. Using (C.9), (C.3), and (C.13) we can re-write (C.7) as

$$\delta \lambda = \delta l \frac{a}{nf} \cos \left(\sin^{-1}\left(\frac{n\lambda_0}{2a \cos \beta}\right) - \beta\right)$$  \hspace{1cm} (C.14)

The pixels on the CCD have $\delta l = 14 \mu m$ which gives a pixel resolution of $0.0076 nm$ at an alignment wavelength of $468.6$ nm. The resolution is weakly dependent on the central wavelength in question
and varies from $0.0073\text{nm}$ at $\lambda_0 = 700\text{nm}$ to $0.0077\text{nm}$ at $\lambda_0 = 400\text{nm}$. Given that alignment adjustments might cause a slight deviation in $f$ and $\beta$, the number given here may have a few percent error and this relation should probably be calibrated.

The preceding discussion covered the effects of diffraction on resolution but did not consider the role entrance optics can play. In particular, the entrance slit of the spectrometer is imaged creates an image that is passed through the optical system. It can be shown (see Section 2.6 of [107]) that reflection off the diffraction grating magnifies or shrinks this image onto the output focal plane, thus limiting resolution. This "Anamorphic" magnification is described by the equation:

$$\frac{b}{a} = \frac{\cos \phi}{\cos \theta}$$  \hfill (C.15)

where $a$ is the width of the entrance slit and $b$ is the width of the image formed on the detector. Using definitions from the previous secton, we find

$$\frac{b}{a} = \frac{\cos \left( \sin^{-1} \left( \frac{n\lambda_0}{2a \cos \beta} \right) - \beta \right)}{\cos \left( \sin^{-1} \left( \frac{n\lambda_0}{2a \cos \beta} \right) + \beta \right)}$$  \hfill (C.16)

This gives $a = \{.93 - .96\}b$ depending on the wavelength. For a 50$\mu$m slit width, the corresponding image will be $\approx 47\mu$m wide and the maximum achievable resolution will be $\approx 0.046\text{nm}$.

To achieve the pixel-limited resolution we would need to use a 15$\mu$m entrance slit. Unfortunately, the CCD is not sensitive enough to measure these low levels of light.

### C.1 Line Broadening by the Linear Stark Effect

The Linear Stark effect is the result of the electric field from charged particle collisions altering the quantum states of bound electrons. Calculations of the statistics of collision rates on exited neutrals show that this process yields a Lorentzian line shape with a full width at half max (FWHM) given by:

$$\text{FWHM} = \alpha n^{2/3}$$  \hfill (C.17)
where $\alpha$ depends on the temperature and the details of the line emission and $n$ is the density of the plasma. See the explanation in Sec 6.4 of [108].

When one process (such as Stark Broadening) produces a Lorentzian line shape and another process (Temperature Broadening) produces a Gaussian line shape, the resulting line will have a shape that is the convolution of the two. This function is called a Voigt profile and can be approximated with a series expansion [109].

To use the Stark Effect to measure density, H$\beta$ line shapes from the CCD detector are fit to a Voigt profile using non-linear least squares. This gives the FWHM of both the Lorentzian and Gaussian components of the Voigt distribution. (In the Line-tied Reconnection Experiment, the Lorentzian FWHM is an order of magnitude larger than the Gaussian FWHM.) Finally, these values are compared to the published tables of [110] to get density. Similar results can be obtained using lines of neutral helium (specifically 4438Å) and the results of [111].

Since the Spectrometer samples a large region of the plasma, there is no way to determine the spatial location of this density measurement.
Appendix D

Interferometer

Inspired by results on the Swarthmore Spheromak Experiment (SSX), a modified Mach-Zender Interferometer is under construction on the Line-tied Reconnection experiment at the University of Wisconsin for time and spatially dependent measurement of plasma density through non-perturbative means. The purpose of this document is to explain the optical and mechanical design of the interferometer and to forecast expected results. While inspiration for this project came from the SSX, it should be noted that the heterodyne modifications to traditional Mach-Zender designs used here were originally proposed by Buchenauer and Jacobson [112].

D.1 Optical Design

A cartoon of the Interferomter setup is given in Fig. D.1. A Helium-Neon Laser produces a 10mw, linearly polarized beam at 632.8 nm. Mirror 1 is included for physical compactness and directs the beam into beam splitter one, creating two optical paths. One of these beams traverses the optical table while the other passes through the plasma and a filter to remove plasma light. Each beam passes through a series of wave plates for polarization reasons, this will be explained in detail in the next section. The beams are recombined in a second beam splitter and passed a Wollastion Prism to split polarization components. Finally, the two polarized components are detected by amplified photo diodes.

The mirrors purchased from Thorlabs are coated to have >99% reflectivity for all polarizations
at an incoming angle of 45°. Any reflection, however, introduces a phase shift of $\pi$.

The beam splitters used have almost identically 50-50 reflection-transmission coefficients. The beam splitters have no preferential polarization for either the reflected or transmitted beams. The reflected beam will, however, pick up a $\pi$ phase shift as in the mirrors.

Uniaxial crystals are (typically crystalline, occasionally plastic) materials where the index of refraction in one direction is different than the indicies of refraction in the other two directions, i.e. $n_x \neq n_y = n_z$. This different axis is called the optical or extraordinary axis. If light is passed through a uniaxial crystal perpendicular to the optical axis, components of the light polarized along the optical axis will receive a different phase shift than components polarized perpendicular to the optical axis. Thus, uniaxial crystals are used to manipulate the polarization of light.

The wave plates used on the interferometer are made from calcite crystals where $n_{\text{extraordinary}} < n_{\text{ordinary}}$. The extraordinary direction in such a wave plate is then called fast, designated $\hat{f}$, and the ordinary direction is called slow, designated $\hat{s}$. The phase shift difference between the two axis going through such a crystal is then $\Phi = \frac{2\pi\delta n L}{\lambda_0}$. The physical length (L) of the crystal then determines the phase shift for a certain wave length.

Half-wave plates are constructed such that $\Phi = \pi$. A beam with linear polarization vector
\( \hat{\rho} = \cos \theta \hat{f} + \sin \theta \hat{s} \) passing through the half-wave plate will pick up a \( \pi \) phase shift on the slow component, giving the outgoing beam a polarization vector of \( \hat{\rho} = \cos \theta \hat{f} - \sin \theta \hat{s} \). Thus, the net effect of a half wave plate is to rotate the incoming polarization vector by an angle \( 2\theta \).

As the name suggests, quarter-wave plates are constructed such that \( \Phi = \pi/2 \). If a linearly polarized beam is introduced with \( \hat{\rho} = \cos \theta \hat{f} + \sin \theta \hat{s} \) the exiting beam will have \( \hat{\rho} = \cos \theta \hat{f} + i \sin \theta \hat{s} \) - thus producing an elliptically polarized beam. If \( \theta = 45^\circ \), the beam is circularly polarized in the left-handed direction. If \( \theta = -45^\circ \), the beam is circularly polarized in the right-handed direction.

![Figure D.2: An illustration of the operating mechanism of a Wollaston prism. The arrows on each crystal indicate the optical axis. Image Credit: Wikipedia.com](image)

The Wollaston Prism is constructed from 2 triangular quartz crystals \( (n_{\text{extraordinary}} > n_{\text{ordinary}}) \) mated together such that the optical axes of the two crystals are perpendicular as seen in Fig. D.2. The net effect of such a mating is that at the interface light polarized along the first optical axis sees a transition from high \( n \) to low \( n \), while the light polarized along the second optical axis sees a transition from low \( n \) to high \( n \). By Fresnel’s equations the two polarization states diverge.

### D.2 Derivation of Detector Output

For the following analysis we shall call the branch of the Interferometer which passes through the plasma S (for scene) and the branch which does not interact with the plasma R (for reference). For polarization purposes we will assume that the x direction is perpendicular to both the beam
propagation and the normal of the interferometer table, while the y direction is perpendicular to the beam propagation and parallel to the table normal. This analysis also assumes that the path lengths of the two beams are identical such that we can ignore the kz component of the wave vector for simplicity. This constraint will be relaxed in Sec D.3. Phase angles will be measured in radians while optical component alignments and polarization will be measured in degrees. Finally, attenuation and differences in beam intensities will be ignored for this analysis. Analysis to be performed at a later date will include them.

For later mathematical ease, we will assume that the laser is linearly polarized at $45^\circ$ from the plane of the interferometer table such that $\vec{E} = e^{i\omega t} \hat{x} + E e^{i\omega t} \hat{y}$, where $\omega = 2\pi f$ is the laser frequency. The electric fields of the two beams leaving the first beam splitter will be

$$\vec{E}_R = -\frac{E}{2} e^{i\omega t} \hat{x} - \frac{E}{2} e^{i\omega t} \hat{y} \tag{D.1}$$

$$\vec{E}_S = \frac{E}{2} e^{i\omega t} \hat{x} + \frac{E}{2} e^{i\omega t} \hat{y} \tag{D.2}$$

where the reference beam has picked up an overall $\pi$ phase shift from the reflection off the beam splitter.

The reference beam is reflected by another mirror ($\pi$ phase shift) and then passes through a quarter wave plate aligned with its fast and slow axis along $\hat{x}$ and $\hat{y}$ respectively. This results in a left-handed circularly polarized beam with

$$\vec{E}_R = \frac{E}{2} e^{i\omega t} \hat{x} + i\frac{E}{2} e^{i\omega t} \hat{y} \tag{D.3}$$

In practice the beam splitter transmits and reflects slightly different amplitudes and phases of the incident polarizations and vacuum chamber windows can slightly polarize the laser beam. To compensate for these small effects the scene beam is passed through a carefully aligned half-wave plate. After reflecting off a mirror (gaining a phase shift of $\pi$) and passing through the plasma the linearly-polarized scene beam gains a phase of $\phi = \frac{\omega}{2n_c} \int n_c dl$, where $n_c = \frac{\rho_{onc} \omega^2}{e^2}$ is the cut-off density for the laser. The S beam passes through a notch filter to remove any contaminating plasma.
radiation and then has electric field:

\[
\vec{E}_S = \frac{E}{2} e^{i\omega t - i\phi} \hat{x} - \frac{E}{2} e^{i\omega t - i\phi} \hat{y}
\]  

(D.4)

The two beams are now recombined in a second beam splitter. Adding (D.3) and (D.4) (with a final \(\pi\) phase shift) gives

\[
\vec{E}_{\text{tot}} = \frac{E}{2} (e^{i\omega t} + e^{i\omega t - i\phi}) \hat{x} + \frac{E}{2} (ie^{i\omega t} + e^{i\omega t - i\phi}) \hat{y}
\]  

(D.5)

The combined beam is then passed through a Wollaston Prism which splits the \(\hat{x}\) and \(\hat{y}\) components of the incoming beam into two beams with a relative divergence of 20° and directs them to the two detectors. Because detector output voltage is proportional to intensity and intensity is proportional to \(|E|^2\), we can calculate the output voltage on the two detectors as

\[
V_1 = k_1 (1 + \cos \phi)
\]

(D.6)

\[
V_2 = k_2 (1 + \sin \phi)
\]

(D.7)

In general, will assume that variations in sensitivity and amplification on the two detectors will mean that \(k_1 \neq k_2\). These values will be calibrated for the interferometer when it is in place.

### D.3 Practical Analysis Formula

For generality we will assume some path length difference in the two arms of the interferometer. This difference will introduce a phase shift of \(\phi_0\) to the problem. Thus, before the shot we will have:

\[
\frac{V_1}{k_1} = (1 + \cos \phi_0)
\]

(D.8)

\[
\frac{V_2}{k_2} = (1 + \sin \phi_0)
\]

(D.9)
Including the plasma phase shift during the shot and performing some algebra, we get:

\[
\frac{V_1}{k_1} = 1 + \cos(\phi_0 + \phi) \quad (D.10)
\]

\[
\frac{V_2}{k_2} = 1 + \sin(\phi_0 + \phi) \quad (D.11)
\]

\[
\frac{\Delta V_1}{k_1} = \cos(\phi_0 + \phi) - \cos \phi_0 \quad (D.12)
\]

\[
\frac{\Delta V_2}{k_2} = \sin(\phi_0 + \phi) - \sin \phi_0 \quad (D.13)
\]

\[
\left( \frac{\Delta V_1}{k_1} \right)^2 + \left( \frac{\Delta V_2}{k_2} \right)^2 = 2(1 - \cos \phi) \quad (D.14)
\]

Finally, this yields the formula for phase shift due to the plasma:

\[
\phi = \cos^{-1} \left( 1 - \frac{1}{2} \left( \frac{\Delta V_1}{k_1} \right)^2 + \left( \frac{\Delta V_2}{k_2} \right)^2 \right) \quad (D.15)
\]

If a 1/4 wave plate is included in the scene beam, it can be show that (D.7) becomes \( V_2 = k_2(1 - \cos \phi) \), loosing the phase direction information, and (D.15) becomes

\[
\phi = \cos^{-1} \left( 1 - \frac{1}{4} \left( \frac{\Delta V_1}{k_1} \right)^2 + \left( \frac{\Delta V_2}{k_2} \right)^2 \right) \quad (D.16)
\]

### D.4 Resolution

To find the minimum density we will be able to measure we need to compare the minimum voltage output we can measure to the relevant phase shift. To do this, assume that the plasma induced phase shift is small and expand (D.12) in complex notation

\[
\frac{\Delta V_1}{k_1} = e^{i\phi_0} e^{i\phi} - e^{i\phi_0}
\]

\[
= \left( 1 + i\phi - \frac{\phi^2}{2} + \ldots \right) e^{i\phi_0} - e^{i\phi_0}
\]

\[
= -\phi \sin \phi_0 - \frac{\phi^2}{2} \cos \phi_0 + \ldots
\]
Where we have gone back to real space for the last step. Similarly for (D.13),

$$\frac{\Delta V_1}{k_1} = -ie^{i\phi_0}e^{i\phi} + ie^{i\phi_0}$$

(D.20)

$$= -i\left(1 + i\phi - \frac{\phi^2}{2} + \ldots\right)e^{i\phi_0} + ie^{i\phi_0}$$

(D.21)

$$= \phi \cos \phi_0 - \frac{\phi^2}{2} \sin \phi_0 + \ldots$$

(D.22)

Because the detectors have an output range of 0-10V, we will assume that $k_1 = k_2 = 5V$. Because the bit resolution of the D-tacqs is .3mv, we will assume a noise floor of 1mv. Now we only need $\phi_0$ to determine the lowest phase angle we can detect. Since the final analysis in Eq. (D.15) depends on the squared sum of $\Delta V_1$ and $\Delta V_2$, we know that $\phi_0 = 0$ and $\phi_0 = \pi$ will yield lower limits than $\phi_0 = \pi/2$. Thus we will use $\phi_0 = \pi/2$ as a worst case scenario. Using the range of detector and digitizer, this yields

$$\phi_{\text{min}} = \frac{\Delta V_{\text{min}}}{\cos \frac{\pi}{2} k_1} = .000282$$

(D.23)

Assuming an effective path length of 12cm, this phase corresponds to a minimum density of $1.3E18 m^{-3}$. This is a theoretical minimum density, the actual achievable density will probably be several times this ($5E18 m^{-3}$?) due to finite alignment errors and sensitivity issues. On the maximum side, a phase shift of $2\pi$ corresponds to a density of $2.9E22 m^{-3}$, well above typical densities in the experiment.
Bibliography


