STABILITY ANALYSIS OF CYLINDRICAL VLASOV EQUILIBRIA

BY

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INTRODUCTION

Plasmas are inherently very complex systems, and any attempt to understand and predict their behavior must include simplifications and approximations. In stability analysis we commonly make two types of approximations: dynamic and geometric. We replace the individual particle equations of motion by an equation of motion for a continuous fluid either in configuration space (MHD and other "fluid" models), or phase space ("kinetic" models such as the Vlasov and Fokker-Planck equations). And we often simplify the geometry by ignoring boundaries and regarding the plasma as filling all space homogeneously or varying only in one linear direction (the "local approximation"[(23)]). There is generally a trade-off between these types of approximation in that the more sophisticated the dynamics, the cruder the geometry, and vice-versa, in order that the resulting equations be solvable. Thus stability analysis of a toroidal system can be carried out with a fair degree of rigor only in a fluid approximation, such as MHD, while velocity space instabilities are calculated by kinetic theory in greatly simplified geometries. Only recently, with the availability of large and fast computers, has it been feasible to treat problems which are both kinetic and non-local.

We describe here a general method of stability analysis which may be applied to a large class of such problems, namely those which are described dynamically by the Vlasov equation, and geometrically by cylindrical symmetry. In Chapter I, we present the method for the simple case of the Vlasov-Poisson (electrostatic) equations, and apply the results to a calculation of the lower-hybrid-drift instability in a plasma with a rigid rotor distribution function, a problem which has been treated by Davidson(7) using a somewhat different method. In Chapter II the method is extended to the full Vlasov-Maxwell (electromagnetic) equations, and in Chapter III we apply these results to a calculation of the instability of the extraordinary electromagnetic mode in a relativistic E-layer interacting with a background plasma. The results of these calculations are compared to those of Striffler and Kammash,(12) who have treated this problem in the local approximation.

One of the most important aspects of the present work is the method of carrying out the integration of the perturbed distribution function over phase space. The approach used here was inspired by the work of Lewis and Symon,(1) who expand the perturbed distribution function in eigenfunctions of the Liouville operator. In Appendix A we show that the two approaches are equivalent.

I. ELECTROSTATIC CASE AND APPLICATION TO LOWER HYBRID DRIFT INSTABILITY

In this chapter, to illustrate the general approach as simply as possible, we consider a cylindrical plasma in the electrostatic approximation. This approximation is often referred to as the "low-beta" limit. However, it has been shown[2,3,4] that a more appropriate characterization of a plasma for which the electrostatic approximation is valid is $\omega_{pe} \ll ck$, where ω_{pe} is the electron plasma frequency, k is a typical wavenumber, and c is the velocity of light.

The cylindrical coordinate system to be used is shown in Fig. (1.1). The plasma column is taken to be infinitely long in the z-direction and azimuthally symmetric. Its axis of symmetry is taken as the z-axis of coordinates, and it is surrounded by a coaxial conducting cylinder of radius R. The purpose of introducing this cylinder is simply to make the radial mode numbers discrete; if we are dealing with a problem in which a conducting boundary plays no significant role, we may recover the continuum modes by taking the limit $R \to \infty$. To make the modes discrete in the z-direction as well, we impose periodic boundary conditions in the z-direction with periodicity length L_z. Again, this restriction is simply for mathematical convenience, and we may remove it by allowing $L_z \to \infty$.

The only non-ignorable coordinate is r, and so the equilibrium scalar and vector potentials are $\phi^\circ(r)$ and $\underline{A}^\circ(r)$, where the 0 denotes the equilibrium value. The equilibrium fields are then:

$$\underline{E}^\circ = E^\circ(r)\hat{r} \ ,$$
$$\underline{B}^\circ = B^\circ_\theta(r) + B^\circ_z(r)\hat{z} \ . \qquad (1.1)$$

The equilibrium distribution functions will be functions of the particle constants of the motion:

$$f_{oj}(\underline{r},\underline{v}) = f_{oj}(H,P_\theta,P_z) \ . \qquad (1.2)$$

Here the subscript "j" denotes particle species, and

$$H = \frac{m_j}{2}(v_r^2 + v_\theta^2 + v_z^2) + e_j\phi^\circ(r) \ , \qquad (1.3)$$

$$P_\theta = m_j r v_\theta + \frac{e_j}{c} A^\circ_z(r) \ , \qquad (1.4)$$

$$P_z = m_j v_z + \frac{e_j}{c} A^\circ_z(r) \ . \qquad (1.5)$$

In the collisionless electrostatic approximation the plasma is described by the Vlasov-Poisson equations:

$$\left[\frac{\partial}{\partial t} + \underline{v}\cdot\nabla + \frac{e_j}{m_j}\left(\underline{E} + \frac{\underline{v}\times\underline{B}}{c}\right)\cdot\frac{\partial}{\partial \underline{v}}\right] f(\underline{r},\underline{v},t) = 0 \ , \qquad (1.6)$$

$$\nabla^2\phi(\underline{r},t) = -\sum_j 4\pi e_j \int d^3v f(\underline{r},\underline{v},t) \ . \qquad (1.7)$$

For purposes of stability analysis we linearize the above equations, writing

$$f(\underline{r},\underline{v},t) = f_o(\underline{r},\underline{v}) + f_1(\underline{r},\underline{v},t) ,$$

$$\phi(\underline{r},t) = \phi_o(\underline{r}) + \phi_1(\underline{r},t) .$$

Here f_1 and ϕ_1 are small perturbations to be added to the quantities of equations (1.1) and (1.2). (Since in the electrostatic approximation $\nabla \times \underline{E} = 0$, the magnetic field and the vector potential $\underline{A}$ are not perturbed.) The linearized equations then read

$$\begin{aligned}(\frac{\partial}{\partial t} + L_o)\, f_1(\underline{r},\underline{v},t) &= - \frac{e_j}{m_j}\, \underline{E}_1(\underline{r},t) \cdot \frac{\partial}{\partial \underline{v}}\, f_{oj}(\underline{r},\underline{v}) \\ &= \frac{e_j}{m_m}\, [\nabla\phi_1(\underline{r},t)] \cdot \frac{\partial}{\partial \underline{v}}\, f_{oj}(\underline{r},\underline{v}) ,\end{aligned} \tag{1.8}$$

$$\nabla^2\phi_1(\underline{r},t) = - \sum_j 4\pi e_j \int d^3v f_{ij}(\underline{r},\underline{v},t) , \tag{1.9}$$

where we have defined the equilibrium Liouville operator

$$L_o = \underline{v} \cdot \nabla + \frac{e_j}{m_j}\, [\underline{E}_o + \frac{1}{c}\, (\underline{v}\times\underline{B}_o)] \cdot \frac{\partial}{\partial \underline{v}} .$$

From (1.3) - (1.5) we have the relations

$$\frac{\partial H}{\partial \underline{v}} = m_j\underline{v} \quad , \quad \frac{\partial P_\theta}{\partial \underline{v}} = m_j r\hat{\theta} \quad , \quad \frac{\partial P_z}{\partial \underline{v}} = m_j\hat{z} .$$

Consequently (1.8) may be rewritten:

$$\left(\frac{\partial}{\partial t} + L_o\right) f_{1j} = e_j \left[\frac{\partial f_{oj}}{\partial H} \underline{v} + \frac{\partial f_{oj}}{\partial P_\theta} r\hat{\theta} + \frac{\partial f_{oj}}{\partial P_z} \hat{z}\right] \cdot \nabla\phi_1$$

$$= e_j \left[\frac{\partial f_{oj}}{\partial H} \underline{v} \cdot \nabla + \frac{\partial f_{oj}}{\partial P_\theta} \frac{\partial}{\partial \theta} + \frac{\partial f_{oj}}{\partial P_z} \frac{\partial}{\partial z}\right] \phi_1 \ . \qquad (1.10)$$

We are interested in unstable modes, so we take the time dependence of all perturbed quantities to be of the form $e^{-i\omega t}$, with $\mathrm{Im}(\omega) > 0$. Thus we write

$$\phi_1(\underline{r},t) = \phi_1(\underline{r})\, e^{-i\omega t} \quad , \quad f_{1j}(\underline{r},\underline{v},t) = f_{1j}(\underline{r},\underline{v})\, e^{-i\omega t} \ .$$

The operator $(\frac{\partial}{\partial t} + L_o)$ in (1.10), acting on a function of phase space variables and time, is equivalent to the total (or comoving) time derivative. It represents the time derivative of the function as seen by a particle moving along an unperturbed trajectory (i.e., the trajectory it would follow in the equilibrium fields $\underline{E}_o$, $\underline{B}_o$ which appear in L_o). Thus we may write

$$\left(\frac{\partial}{\partial t} + L_o\right)[f_{1j}(\underline{r},\underline{v})\, e^{-i\omega t}] = \frac{d}{dt} \{f_{1j}[\underline{r}(t),\underline{v}(t)]\, e^{-i\omega t}\} \ , \qquad (1.11)$$

where $\underline{r}(t)$, $\underline{v}(t)$ represent the unperturbed particle orbits. We may solve (1.10) for the first order distribution function by integrating over time:

$$f_{1j}(\underline{r},\underline{v})\, e^{-i\omega t} = \int_{-\infty}^{t} dt' \frac{d}{dt'} \{f_{1j}[\underline{r}'(t'),\underline{v}'(t')]\, e^{-i\omega t'}\}$$

$$= e_j \int_{-\infty}^{t} dt'\, e^{-i\omega t'} \left[\frac{\partial f_{oj}}{\partial H} \frac{d}{dt'} + \frac{\partial f_{oj}}{\partial P_\theta} \frac{\partial}{\partial \theta'} + \frac{\partial f_{oj}}{\partial P_z} \frac{\partial}{\partial z'}\right] \phi_1[\underline{r}'(t')] \ , \qquad (1.12)$$

where $\underline{r}'(t')$, $\underline{v}'(t')$ represent the unperturbed trajectory of a particle as a function of the dummy variable t' and

$$\underline{r}'(t) = \underline{r} \quad , \underline{v}'(t) = \underline{v} \ . \tag{1.13}$$

In obtaining Eq. (1.12), we have used the fact that $\underline{v}' \cdot \nabla = \frac{d}{dt'}$ when acting on a function of $\underline{r}'$.

As we have assumed periodicity in the z-direction with periodicity length L_z, we may resolve $\phi_1(\underline{r})$ into its Fourier components in the ignorable coordinates θ and z:

$$\phi_1(\underline{r}) = \sum_{\ell,k} \phi_{\ell,k}(r) \ e^{i(\ell\theta+kz)} \ , \tag{1.14}$$

where $k = \frac{2\pi n}{L_z}$, n an integer.

Using (1.12) and (1.13), Poisson's equation (1.7) becomes

$$\left(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}\right) \sum_{\ell,k} \phi_{\ell,k} (r) \ e^{i(\ell\theta+kz)}$$

$$= - \sum_j 4\pi e_j^2 \int d^3v \int_{-\infty}^{t} dt' \ e^{-i\omega(t'-t)}$$

$$\cdot \left[\frac{\partial f_{oj}}{\partial H}\frac{d}{dt'} + \frac{\partial f_{oj}}{\partial P_\theta}\frac{\partial}{\partial\theta'} + \frac{\partial f_{oj}}{\partial P_z}\frac{\partial}{\partial z'}\right]$$

$$\cdot \sum_{\ell' k'} \phi_{\ell' k'} (r') \ e^{i(\ell'\theta'+k'z')} \ . \tag{1.15}$$

Note that though the right side of (1.15) contains t, it is actually independent of the value of t. In fact, we could remove t altogether by defining a new variable $\tau = t' - t$ and replacing $\int_{-\infty}^{t} dt'$ by $\int_{-\infty}^{0} d\tau$. We retain the formal t "dependence", however, as it will be useful below.

To isolate one Fourier component on the left side of (1.15), we multiply by $\frac{1}{2\pi} e^{-i(\ell\theta+kz)}$ and integrate over θ and z. From Eq. (1.13), we see that the quantities $\theta' - \theta$ and $z' - z$ are independent of θ and z, respectively, since for fixed t' and t changing θ changes θ' by the same amount, and similarly for z. Using the identities

$$k'z' - kz = k'(z'-z) + (k'-k)z \ ,$$

$$\ell'\theta' - \ell\theta = \ell'(\theta'-\theta) + (\ell'-\ell)\theta \ ,$$

we obtain

$$\left(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} - \frac{\ell^2}{r^2} - k^2\right) \phi_{\ell,k}(r) =$$

$$- \sum_j 4\pi e_j^2 \int d^3v \int_{-\infty}^{t} dt' e^{-i\omega(t'-t)} \left[\frac{\partial f_{oj}}{\partial H} \frac{d}{dt'} + \frac{\partial f_{oj}}{\partial P_\theta} \frac{\partial}{\partial \theta'} + \frac{\partial f_{oj}}{\partial P_z} \frac{\partial}{\partial z'}\right]$$

$$\cdot \ \phi_{\ell,k}(r') \, e^{i[\ell(\theta'-\theta)+k(z'-z)]} \ . \qquad (1.16)$$

Note from the above argument that the right side of (1.16) does not depend on the ignorable coordinates, even though they appear there.

Next, we wish to expand the radial dependence of the perturbed potential in eigenfunctions of the Laplacian:

$$\phi_{\ell,k}(r) = \sum_n \alpha_n \phi_n(r) \ , \qquad (1.17)$$

where $\phi_n(r)$ satisfies the eigenvalue equation

$$\left(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} - \frac{\ell^2}{r^2}\right) \phi_n(r) = -\lambda_n^2 \phi_n(r) \ . \qquad (1.18)$$

Here we have suppressed the subscripts ℓ and k on ϕ_n. Thus we have

$$\phi_n(r) = A_n J_\ell(\lambda_n r) \ ,$$

where J_l is the ℓ^{th} order Bessel function of the first kind,

$$A_n = \frac{\sqrt{2}}{RJ_{\ell+1}(\lambda_n R)}$$

is a normalization constant, and λ_n is the n^{th} root of the equation $J_\ell(\lambda_n R) = 0$. The functions $\phi_n(r)$ then satisfy the orthonormality relation

$$\int_0^R drr\phi_{n'}(r)\ \phi_n(r) = \delta_{nn'} \ . \qquad (1.19)$$

Substituting (1.18) into (1.16), multiplying by $r\phi_{n'}(r)$, and integrating over r yields

$$\sum_n \alpha_n(-\lambda_n^2-k^2)\ \delta_{nn'} = -\sum_j 4\pi e_j^2 \int_0^R drr\phi_{n'}(r) \int d^3v \int_{-\infty}^t dt'\ e^{-i\omega(t'-t)}$$

$$\left[\frac{\partial f_{oj}}{\partial H}\frac{d}{dt'} + \frac{\partial f_{oj}}{\partial P_\theta}\frac{\partial}{\partial \theta'} + \frac{\partial f_{oj}}{\partial P_z}\frac{\partial}{\partial z'}\right]$$

$$\cdot\ e^{i[\ell(\theta'-\theta)+k(z'-z)]} \sum_n \alpha_n \phi_n(r') \ . \qquad (1.20)$$

This equation is a linear relation in the expansion coefficients

α_n, and we can write it as

$$\sum_n D_{nn'}(\omega)(\lambda_n^2+k^2)\ \alpha_{n'} = 0 ,$$

where

$$D_{nn'}(\omega) = \delta_{nn'} - \sum_j \frac{4\pi e_j^2}{k_n^2} \int_o^R drr \int d^3v\phi_n(r)\ e^{-i(\ell\theta+kz-\omega t)}$$

$$\cdot \int_{-\infty}^t dt'\ e^{-i\omega t'}\ [\frac{\partial f_{oj}}{\partial H} \frac{d}{dt'} + i\ell \frac{\partial f_{oj}}{\partial P_\theta} + ik_z \frac{\partial f_{oj}}{\partial P_z}]$$

$$\cdot\ \phi_{n'}(r')\ e^{i(\ell\theta'+kz')} \qquad (1.21)$$

and

$$k_n^2 \equiv \lambda_n^2 + k^2 .$$

To carry out the time integral in (1.21), we must determine the unperturbed orbits $\underline{r}'(t')$. The particle Hamiltonian does not depend on θ, z, or t, so H is a constant equal to the particle energy and we have

$$H(r,P_r,P_\theta,P_z) = E .$$

Since P_θ, P_z, and E are constants of the motion, we can solve for P_r as a function of r:

$$P_r(r) = P_r(r,E,P_\theta,P_z) .$$

This gives $\dot{r}$ as a function of r:

$$\dot{r} = \frac{\partial H}{\partial P_r} = \dot{r}(r) \ .$$

If the particle motion in r is bounded and non-asymptotic, then a particle which is at r_o at some time t must return to r_o at some latter time t + T. Since the Hamiltonian depends only on r, when the particle returns to r_o, it must have the same radial velocity $\dot{r}(r_o)$. Thus the motion in r is periodic with period $T(P_\theta, P_z, E)$. Now $\dot{\theta} = \frac{\partial H}{\partial P_\theta}$ depends on t only through r so it too must be periodic. Thus we can write

$$\theta(t) = \eta t + \tilde{\theta}(t) + \theta_o \ ,$$

where η is a constant, $\tilde{\theta}(r)$ is periodic with period T, and θ_o is an initial value chosen so $\tilde{\theta}(0) = 0$. Similarly, we have $z(t) = \sigma t + \tilde{z}(t) + z_o$, where σ is a constant, $\tilde{z}(t)$ is periodic with period T, and z_o is chosen so $\tilde{z}(t = 0) = 0$. Consequently we may write

$$\phi_n(r')e^{i(\ell\theta'+kz')} = A_n \ e^{i(\ell\eta+k\sigma)} + J_\ell(\lambda_n r')e^{i\ell\tilde{\theta}'} \ e^{ik\tilde{z}'} \ e^{i(\ell\theta_o+kz_o)} \ . \tag{1.22}$$

In order to perform the time integration in (1.21), we wish to express the function in (1.22) as a Fourier series in time. First consider the z term $e^{ik\tilde{z}'(t')}$. Since $\tilde{z}$ is periodic in time, we may write

$$\tilde{z}(t) = \sum_n b_n \sin n\Omega t \quad ; \quad \Omega = \frac{2\pi}{T(E,P_\theta,P_z)} \quad . \tag{1.23}$$

Using the identity

$$e^{ip\sin q} = \sum_{n=-\infty}^{\infty} J_n(p)\, e^{inq}$$

we have

$$e^{ik\tilde{z}(t)} = \prod_n e^{ikb_n \sin n\Omega t} = \prod_n (\sum_m J_m(kb_n)\, e^{imn\Omega t}) \ . \tag{1.24}$$

Thus we have expanded the function $e^{ik\tilde{z}(t)}$ in a Fourier series in time. Next we wish to represent $J_\ell(\lambda_n r)\, e^{i\ell\tilde{\theta}}$ in this form as well. Consider first the function $re^{i\tilde{\theta}}$; it is periodic, and so we can write it as a Fourier series:

$$r(t)\, e^{i\tilde{\theta}(t)} = \sum_n a_n\, e^{i(\Omega_n t+\gamma_n)} \ , \tag{1.25}$$

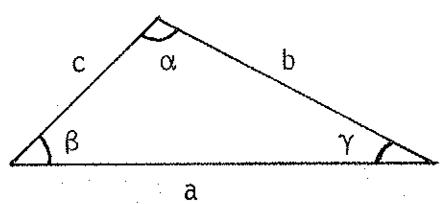
where the a_n are chosen real, γ_n is a phase factor, and $\Omega_n = m_n\Omega$, m_n an integer and Ω given in Eq. (1.23). Here m_n is taken to be the frequency of the n^{th} largest term, so that the largest terms in the series occur first. In other words, instead of ordering the terms in the Fourier series by their frequency as is normally done, we order them by the magnitude of their coefficients. This is done solely for heuristic reasons, to simplify the representation of the particle orbits given below. To illustrate the approach, suppose we wish to take four terms of (1.24) as an adequate approximation to the particle motion. Then

$$re^{i\theta} = a_1\, e^{i(\Omega_1 t+\gamma_1)} + a_2\, e^{i(\Omega_2 t+\gamma_2)}$$

$$+ a_3\, e^{i(\Omega_3 t+\gamma_3)} + a_4\, e^{i(\Omega_4 t+\gamma_4)} \quad .$$

This function can be represented in the complex plane as in Fig. (1.2). Note that it corresponds to the particle motion in the x-y plane with the constant precession $e^{i\eta t}$ factored out.

Next we make use of Graf's theorem,[5] an addition theorem for Bessel functions, which states that for any triangle



we have

$$J_p(c)\ e^{ip\beta} = \sum_{m=-\infty}^{\infty} J_{p+m}(a)\ J_m(b)\ e^{im\gamma} \ . \qquad (1.26)$$

We apply this theorem repeatedly to the triangles in Fig. (1.2), obtaining

$$J_\ell(\lambda_r)\ e^{i\ell\tilde{\theta}} =$$

$$e^{i\ell(\Omega_1 t+\gamma_1)} \sum_{m_1} J_{\ell+m_1}(\lambda a_1)\ J_{m_1}(\lambda e) e^{-im_1[\alpha_1+(\Omega_2-\Omega_1)t+\gamma_2-\gamma_1]}(-1)^{m_1} \ ,$$

$$J_{m_1}(\lambda e)\, e^{im_1\alpha_1} =$$

$$\sum_{m_2} J_{m_1+m_2}(\lambda a_2)\, J_{m_2}(\lambda f)\, e^{im_2[\alpha_2+(\Omega_3-\Omega_2)t+\gamma_3-\gamma_2]}\, (-1)^{m_2} ,$$

and finally

$$J_{m_2}(\lambda f)\, e^{im_2\alpha_2} =$$

$$\sum_{m_3} J_{m_2+m_3}(\lambda a_3)\, J_{m_3}(\lambda a_4)\, e^{-im_3[(\Omega_4-\Omega_3)t+\gamma_4-\gamma_3]}\, (-1)^{m_3} .$$

Combining these expressions we have

$$J_\ell(\lambda r)\, e^{i\ell\tilde{\theta}} =$$

$$\sum_{m_1,m_2,m_3} J_{\ell+m_1}(\lambda a_1)\, J_{m_1+m_2}(\lambda a_2)\, J_{m_2+m_3}(\lambda a_3)\, J_{m_3}(\lambda a_4)(-1)^{m_1+m_2+m_3}$$

$$e^{i[(\ell+m_1)(\Omega_1 t+\gamma_1)-(m_1+m_2)(\Omega_2 t+\gamma_2)+(m_2+m_3)(\Omega_3 t+\gamma_3)-m_3(\Omega_4 t+\gamma_4)]} . \tag{1.27}$$

Using (1.24) and (1.27), we can now write

$$\phi_n(r)\, e^{i(\ell\theta+kz)} = \sum_m G_m(E,P_\theta,P_z)\, e^{i(\ell\eta+k\sigma+m\Omega)t} , \tag{1.28}$$

where the coefficients G_m depend on the constants of the motion H, P_θ, and P_z through the coefficients a_n of (1.27) and b_n of (1.24). The best way to determine these coefficients, and the number needed for a good approximation to the particle orbit, will depend on the problem at hand. For the simple field configuration treated below, they may be found analytically; for a more complex equilibrium, it may be easiest to use a Galerkin method, substituting a certain number of terms of the series (1.24) and (1.27) into the equations of motion and solving the resulting algebraic equations for the a_n and b_n. In most cases of practical interest we would expect to need only a few epicycles, as in Fig. (1.2), to represent the particle motion well enough for a linearized stability analysis.

Using the expression (1.28) for the cylindrical harmonics we may rewrite (1.21) as

$$D_{nn'}(\omega) = \delta_{nn'} - \sum_j \frac{4\pi e_j^2}{k_n^2} \int_0^R drr \int d^3v\phi_n(r)\, e^{-i(\ell\theta+kz-\omega t)} \int_{-\infty}^{t} dt'\, e^{-i\omega t'}$$

$$\cdot\, i\left[(\ell\eta+k\sigma+m\Omega)\frac{\partial f_{oj}}{\partial H} + \ell\,\frac{\partial f_{oj}}{\partial P_\theta} + k\,\frac{\partial f_{oj}}{\partial P_z}\right] \sum_m G_m\, e^{i(\ell\eta+k\sigma+m\Omega)t'} \;.$$

Note that θ_o and z_o cancel out of the exponentials due to the fact that the dispersion matrix is diagonal in ℓ and k. Remembering that $\mathrm{Im}(\omega) > 0$, we can perform the t' integration:

$$D_{nn'}(\omega) = \delta_{nn'} - \sum_j \frac{4\pi e_j^2}{k_n^2} \int_0^R drr \int d^3v\phi_n(r)\, e^{-i(\ell\theta+kz)}$$

$$\cdot\, i\left[(\ell\eta+k_z\sigma + m\Omega)\,\frac{\partial f_{oj}}{\partial H} + \ell\,\frac{\partial f_{oj}}{\partial P_\theta} + k\,\frac{\partial f_{oj}}{\partial P_z}\right]$$

$$\cdot\, \sum_m G_m\, e^{i(\ell\eta+k\sigma+m\Omega)t}/i[\ell\eta+k\sigma+m\Omega-\omega] \;. \qquad (1.29)$$

Next we make use of the t "dependence" of the integrand in (1.29).

Writing the integrand as $I(r,\underline{v})$ we have

$$I(r,\underline{v}) = I(r,\underline{v},t) = I(r(t), \underline{v}(t))$$

where in the last equality $r(t)$ and $\underline{v}(t)$ represent the trajectory of a particle as a function of time. Varying t thus generates a curve in phase space along which I is invariant. We now use this fact to carry out part of the integration of I over the phase space variables. First we transform the variables of integration from the velocities to the canonical momenta:

$$\left|\frac{\partial(P_r,P_\theta,P_z)}{\partial(v_r,v_\theta,v_z)}\right| = m_j^3 r \rightarrow rdrd^3v = \frac{1}{m_j^3} drdP_r dP_\theta dP_z \ . \qquad (1.30)$$

Next we wish to change the integration variables r, P_r to t, H.

$$dH = -\dot{P}_r dr + \dot{r} dP_r \quad , \quad dt = \frac{1}{\dot{r}} dt \ ,$$

so

$$\left|\frac{\partial(t,H)}{\partial(r,P_r)}\right| = 1 \quad , \quad drdP_r = dtdH \ . \qquad (1.31)$$

From (1.30) and (1.31) we have

$$rdrd^3v = m_j^{-3} \, dtdHdP_\theta dP_z \ ,$$

so using (1.28), we can write (1.29) as

$$D_{nn'}(\omega) = \delta_{nn'} - \sum_j \frac{4\pi e_j^2}{k_n^2} \int dHdP_\theta dP_z \int_0^T dt \sum_{m'} G_{m'}^* \, e^{-i(\ell\eta+k\sigma+m'\Omega)t}$$

$$\cdot \left[(\ell\eta+k\sigma+m\Omega) \frac{\partial f_{oj}}{\partial H} + \ell \frac{\partial f_{oj}}{\partial P_\theta} + k \frac{\partial f_{oj}}{\partial P_z}\right] \sum_m \frac{G_m e^{i(\ell\eta+k\sigma+m\Omega)t}}{\ell\eta+k\sigma+m\Omega-\omega}$$

$$= \delta_{nn'} - \sum_j \frac{4\pi e_j^2}{k_n^2} \int dHdP_\theta dP_z T\,(H,P_\theta,P_z)$$

$$\cdot \sum_m \left[(\ell\eta+k\sigma+m\Omega) \frac{\partial f_{oj}}{\partial H} + \ell \frac{\partial f_{oj}}{\partial P_\theta} + k \frac{\partial f_{oj}}{\partial P_z}\right] \frac{|G_m(H,P_\theta,P_z)|^2}{\ell\eta+k\sigma+m\Omega-\omega} \quad . \tag{1.32}$$

Now we are left with integrals over the constants of the motion H, P_θ, P_z to perform in order to evaluate the elements of the dispersion matrix D. These must usually be carried out numerically. The normal modes of the system are found by solving

$$\det[\bar{D}(\omega)] = 0$$

for ω, where $\bar{D}$ represents a suitable truncation of D (i.e., one that includes enough of the expansion functions ϕ_n to represent the normal modes to the desired accuracy). If ω has a positive imaginary part, the corresponding mode is unstable, and the imaginary part of ω gives its growth rate.

To illustrate this procedure further, we next consider a simple class of problems for which the series (1.24) and (1.27) have only one term. We take the equilibrium magnetic field to be uniform and directed in the z-direction $\underline{B}_0 = B_0\hat{z}$, and the electric field to be radial and proportional to r: $\underline{E}_0 = E_0 r\hat{r}$, where B_0 and E_0 are constants. These fields give the equilibrium bulk motion of the plasma a "rigid-rotor" character, as explained below. The equation of motion for a particle in these fields is

$$m_m \frac{d\underline{v}}{dt} = e_j E_0 r\hat{r} + \frac{e_j}{c}\,\underline{v} \times B_0\hat{z} \ ,$$

or, in Cartesian coordinates

$$m_m\ddot{x} = e_m E_0 x + \frac{e_j B_0}{c}\,\dot{y} \quad , \quad m_j\ddot{y} = e_j E_0 y - \frac{e_j B_0}{c}\,\dot{x} \quad , \quad \ddot{z} = 0 \ .$$

If we let $\xi = x + iy$, we can write these equations as[6]

$$\ddot{\xi} + i\omega_{cj}\dot{\xi} - \frac{e_j}{m_j} E_0\xi = 0 \ , \qquad (1.33)$$

where $\omega_{cj} = \frac{e_j B_0}{m_j c}$, the cyclotron frequency for species j. This is a second order linear equation, and the solution may be written

$$\xi = ae^{i\omega_a t} + be^{i\omega_b t} \ , \qquad (1.34)$$

where we may take a and b to be real and non-negative, so that

$$r(t=0) = a + b \quad , \quad y(t=0) = 0 \ .$$

Substituting (1.34) into (1.33), we determine ω_a and ω_b from the resulting algebraic equation:

$$\omega_{\substack{a\\b}} = -\frac{\omega_{cj}}{2} \pm \frac{1}{2}\sqrt{\omega_{cj}^2 - \frac{4e_j E_o}{m_j}} \quad . \tag{1.35}$$

For the particle motion to be bounded, we must impose the condition

$$\omega_{cj}^2 \geq \frac{4e_j E_o}{m_j} \quad .$$

Note that $\omega_a \geq \omega_b$.

The coefficients a and b in (1.33) determine the particle's radial position and its velocity, so they may be written in terms of H and P_θ. In practice, as shown below, it is more convenient to express H and P_θ in terms of a and b. In this equilibrium $\dot{z}$ is a constant of the motion, so $\tilde{z} = 0$ in (1.22) and $\sigma = \dot{z}$.

The particle motion in the x-y plane can be represented as in Fig. (1.3). Applying Graf's theorem (1.26) to the triangle (r, a, b), we have

$$\begin{aligned}\phi_n(r)\, e^{i\ell\theta} &= A_n\, e^{i\ell\omega_b t}\, J_\ell(\lambda_n r)\, e^{i\ell\tilde{\theta}} \\ &= A_n\, e^{i\ell\omega_b t} \sum_{m=-\infty}^{\infty} (-1)^m\, J_{\ell+m}(\lambda_n b)\, J_m(\lambda_n a)\, e^{-im(\omega_a - \omega_b)t} \quad .\end{aligned} \tag{1.36}$$

Substituting this expression into (1.21) and writing

$$D_{nn'}(\omega) = \delta_{nn'} + \sum_j \chi^j_{nn'}(\omega)$$

we have

$$\chi^j_{nn'}(\omega) = -\frac{4\pi e_j^2}{m_j k_n^2} A_n A_{n'} \int_o^R drr \int d^3v \sum_{m'=-\infty}^{\infty} (-1)^{m'} J_{\ell+m'}(\lambda_n b)\, J_m(\lambda_n a)$$

$$\cdot\, e^{-i[\ell\omega_b + kv_z - m'(\omega_a-\omega_b)-\omega]t} \int_{-\infty}^{t} dt' \sum_{m=-\infty}^{\infty}$$

$$\cdot \left\{ m_j \frac{\partial f_{oj}}{\partial H_\perp} [i\ell\omega_b - im(\omega_a-\omega_b)] + i\ell m_j \frac{\partial f_{oj}}{\partial P_\theta} + ik \frac{\partial f_{oj}}{\partial v_z} \right\}$$

$$(-1)^m J_{\ell+m}(\lambda_{n'} b)\, J_m(\lambda_{n'} a)\, e^{i[\ell\omega_b + kv_z - m(\omega_a-\omega_b)-\omega]t'} ,$$

where we have defined

$$H_\perp = \frac{m_j}{2}(v_r^2+v_\theta^2) + e_j\phi_o(r)$$

and replaced P_z by v_z using $P_z = m_j v_z$. We will refer to the χ's as "susceptibilities", since they contain the response of the plasma to the field. Since z is an ignorable coordinate, $H_\perp$ and v_z are also constants of the motion.

Doing the integral over t' we obtain

$$\chi^j_{nn'}(\omega) = -\frac{4\pi e_j^2}{m_j k_n^2} A_n A_{n'} \int_o^R drr \int d^3v \sum_{m'=-\infty}^{\infty} (-1)^{m'} J_{\ell+m'}(\lambda_n b)\, J_m(\lambda_n a)$$

$$\cdot\, e^{-i[\ell\omega_b + kv_z - m'(\omega_a-\omega_b)-\omega]t} \sum_{m=-\infty}^{\infty} \left\{ \frac{\partial f_{oj}}{\partial H_\perp} m_j[\ell\omega_b - m(\omega_a-\omega_b)]t + \ell m_j \right.$$

$$\cdot \left. \frac{\partial f_{oj}}{\partial P_\theta} + k\frac{\partial f_{oj}}{\partial v_z} \right\} (-1)^m J_{\ell+m}(\lambda_n b)\, J_m(\lambda_n a)\, \frac{e^{i[\ell\omega_b + kv_z - m(\omega_z-\omega_b)-\omega]t}}{\ell\omega_b + kv_z - m(\omega_s-\omega_b)-\omega} . \tag{1.37}$$

Using (1.30) and (1.31) we change the variables of integration from r, $\underline{v}$ to t, $H_\perp$, P_θ, and v_z and do the resulting integral over t to get

$$\chi^j_{nn'}(\omega) = \frac{4\pi e_j^2}{m_j^3 k_n^2} A_n A_{n'} \left[\frac{2\pi}{\omega_a - \omega_b}\right] \sum_m \int dH_\perp dP_\theta dv_z$$

$$\cdot \left\{ m_j \frac{\partial f_{oj}}{\partial H_\perp} [\ell\omega_b - m(\omega_a - \omega_b)] + \ell m_j \frac{\partial f_{oj}}{\partial v_z} + k \frac{\partial f_{oj}}{\partial v_z} \right\}$$

$$\cdot \frac{J_{\ell+m}(\lambda_n b) J_m(\lambda_n a) J_{\ell+m}(\lambda_{n'} b) J_m(\lambda_{n'} a)}{\omega + m(\omega_a - \omega_b) - kv_z - \ell\omega_b} , \quad (1.38)$$

where $\frac{2\pi}{\omega_a - \omega_b}$ appears as the period T of the motion in r.

For this equilibrium we have

$$H_\perp = \frac{m_j}{2}(v_r^2 + v_\theta^2) - \frac{1}{2} e_j E_o r^2 \quad , \quad P_\theta = m_j r v_\theta + \frac{1}{2} m_j \omega_{cj} r^2 \ .$$

When r = a + b, the particle is at its maximum in r, so that $v_r = 0$, $\theta = 0$, and $v = \omega_a a + \omega_b b$. Thus

$$H_\perp = \frac{1}{2} m_j (\omega_a a + \omega_b b)^2 - \frac{1}{2} e_j E_o (a+b)^2 \quad (1.39)$$

$$P_\theta = m_m (a+b)(\omega_a a + \omega_b b) + \frac{1}{2} m_j \omega_{cj} (a+b)^2 \ . \quad (1.40)$$

Equations (1.36) and (1.37) could be solved for a and b in terms of $H_\perp$ and P_θ and the results substituted into (1.35). However, the expressions for a and b in terms of $H_\perp$ and P_θ are considerably more complicated than (1.36) and (1.37), so it is more convenient to use

(1.36) and (1.37) to convert the $H_\perp$ and P_θ integrals in (1.35) to integrals over a and b. We have

$$\left|\frac{\partial(H_\perp,P_\theta)}{\partial(a,b)}\right| = m_j^2(\omega_a-\omega_b)\left|(\omega_a a+\omega_b b)^2 + (\omega_a a+\omega_b b)\omega_{cj}(a+b)+ \frac{e_j\varepsilon_o}{m_j}(a+b)^2\right| . \tag{1.41}$$

We shall now demonstrate that the expression inside the absolute value signs on the right of (1.41) is always negative, so that we may replace the absolute value signs with a factor of -1. Since $\omega_a - \omega_b > 0$, the expression can be positive only if

$$z^2 + \omega_{cj}z + \frac{e_jE_o}{m_j} \geq 0 , \tag{1.42}$$

where

$$z = \frac{\omega_a a+\omega_b b}{a+b} .$$

If we take the equality in (1.42), we have the same algebraic equation for z as that from which ω_a and ω_b were determined in Eqs. (1.33) - (1.35). Thus

$$z^2 + \omega_{cj}z + \frac{e_jE_o}{m_j} = 0 \quad \text{iff} \quad z = \omega_a \quad \text{or} \quad z = \omega_b ,$$

and the expression on the right of (1.42) can change sign only if z passes through one of the values ω_a or ω_b. But since a + b > 0, $\omega_a - \omega_b \geq 0$, we have

$$\omega_b \le \frac{\omega_a a+\omega_b b}{a+b} \le \omega_a \quad , \quad \omega_b \le z \le \omega_a \ . \tag{1.43}$$

Since the left side of (1.42) is clearly positive for $z \to \pm\infty$, and must change sign at $z = \omega_a$ or $z = \omega_b$, we see that for a in the range of (1.43) the left side of (1.42) must be non-positive. Therefore

$$\left|\frac{\partial(H_\perp,P_\theta)}{\partial(a,b)}\right| = -m_j^2(\omega_a-\omega_b)[(\omega_a a+\omega_b b)^2 + (\omega_a a+\omega_b b)\omega_{cj}(a+b)+ \frac{e_j E_o}{m_j} (a+b)^2]. \tag{1.44}$$

Using (1.44), (1.38) now becomes

$$\chi^j_{nn'}(\omega) = - \frac{8\pi^2 e_j^2}{m_j k_n^2} A_n A_{n'} \sum_{m=-\infty}^{\infty} \int_0^R da \int_0^{R-a} \int_{-\infty}^{\infty} dv_z$$

$$\cdot \left\{m_j \frac{\partial fo_j}{\partial H_\perp} [\ell\omega_b - m(\omega_a-\omega_b)] + \ell m_j \frac{\partial f_{oj}}{\partial P_\theta} + k_z \frac{\partial f_{oj}}{\partial v_z}\right\}$$

$$\cdot [(\omega_a a+\omega_b b)^2 + (\omega_a a+\omega_b b)\ \omega_{cj}(a+b) + \frac{e_j E_o}{m_j} (a+b)^2]$$

$$\cdot \frac{J_{\ell+m}(\lambda_n b)J_m(\lambda_n a)J_{\ell+m}(\lambda_{n'} b)J_m(\lambda_{n'} a)}{\omega+m(\omega_a-\omega_b)-k_z v_z-\ell\omega_b} \ . \tag{1.45}$$

The integrals over a and b in (1.45) have been taken over the range $a + b < R$ so that all orbits considered lie entirely within the cylinder of radius R. Strictly speaking, we can only allow equilibrium distribution functions f_{oj} which vanish identically outside the cylinder. This is because orbits extending outside R will not be treated correctly by the integration over unperturbed orbits, since the expansion of the perturbed potential is not valid there. In many cases, however, it is most convenient to choose a simple analytic function for f_{oj} and allow the limits of the phase space integration to exclude the unwanted orbits. This is more easily accomplished when the matrix elements are written in the form (1.45) than in the form (1.38), since the region in $H_\perp$, P_θ space which corresponds to orbits lying entirely inside the cylinder is much more complicated in structure than the simple triangle in a, b space which is integrated over in (1.45).

Having calculated the susceptibilities $\chi^j_{nn'}(\omega)$ in (1.45), the stability analysis is completed by solving the equation

$$\det[\delta_{nn'} + \sum_j \chi^j_{nn'}(\omega)] = 0$$

for ω using a suitable truncation of the dispersion matrix. How to determine a suitable truncation will be discussed further in Chapter III. The roots ω of this equation with positive imaginary part γ correspond to the unstable modes of the plasma, with γ being the growth rate.

R. C. Davidson[7] has presented a similar method of stability analysis for the special case when f_{oj} is a rigid rotor equilibrium. A rigid rotor equilibrium is characterized by a distribution function of the form

$$f_{oj}(\underline{r}\ \underline{v}) = f_{oj}(H_\perp - \omega_j P_\theta, v_z) ,$$

i.e., f_{oj} depends on $H_\perp$ and P_θ only through the linear combination $H_\perp - \omega_j P_\theta$, where ω_j is a constant for each species. It can be shown that in a reference frame rotating about the z-axis with frequency ω_j the rigid rotor distribution is isotropic in velocity space. Thus in the multi-fluid limit each species appears to rotate in bulk with angular frequency ω_j, and this gives rise to the name "rigid rotor."

For a rigid rotor distribution function we have

$$\frac{\partial f_{oj}}{\partial P_\theta} = -\omega_j \frac{\partial f_{oj}}{\partial H_\perp} ,$$

and (1.38) becomes

$$\chi^j_{nn'}(\omega) = -\frac{4\pi e_j^2}{m_j^3 k_n^2} A_n A_{n'} \frac{2\pi}{\omega_a - \omega_b} \sum_{m=-\infty}^{\infty} \int dH_\perp \int dP_\theta \int dv_z$$

$$\cdot \left\{ m_j \frac{\partial f_{oj}}{\partial H_\perp} [\ell(\omega_b - \omega_j) - m(\omega_a - \omega_b)] + k \frac{\partial f_{oj}}{\partial v_z} \right\}$$

$$\cdot \frac{J_{\ell+m}(\lambda_n b) J_m(\lambda_n a) J_{\ell+m}(\lambda_{n'} b) J_m(\lambda_n a)}{\omega + m(\omega_a - \omega_b) - k v_z - \ell \omega_b} . \qquad (1.46)$$

Defining

$$V_x = v_x + \omega_j y \quad , \quad V_y = v_y - \omega_j x \quad , \quad V_\perp^2 = V_x^2 + V_y^2 \quad , \quad \omega_j^\pm = \omega_{\substack{a \\ b}} ,$$

we may write Davidson's result for the same problem as

$$\chi^j_{nn'}(\omega) = -\frac{4\pi e_j^2}{m_j k_n^2} A_n A_{n'} \int_0^R drr\, J_\ell(\lambda_n r)\, J_\ell(\lambda_{n'} r) \int d^3V \frac{1}{V_\perp} \frac{\partial f_{oj}}{\partial V_\perp}$$

$$+ \frac{4\pi e_j^2}{m_j k_n^2} A_n A_{n'} \int_0^R drr \sum_{p=-\infty}^{\infty} J_\ell(\lambda_n r) J_p\left(\frac{\omega_j^- - \omega_j^-}{\omega_j^+ - \omega_j^-} \lambda_{n'} r\right) J_{\ell-p}\left(\frac{\omega_j^+ - \omega_j}{\omega_j^+ - \omega_j^-} \lambda_n r\right)$$

$$\cdot \int d^3V \sum_{m=-\infty}^{\infty} J_m^2\left(\frac{\lambda_n V_\perp}{\omega_j^+ - \omega_j^-}\right) \frac{\left[k\left(\frac{\partial}{\partial V_z} - \frac{V_z}{V_\perp}\frac{\partial}{\partial V_\perp}\right) + \frac{\omega - \ell\omega_j}{V_\perp}\frac{\partial}{\partial V_\perp}\right] f_{oj}}{\omega - \ell\omega_j^- - (p+m)(\omega_j^+ - \omega_j^-) - kV_z} \,. \tag{1.47}$$

Note that (1.46) is somewhat simpler than (1.47) because in the derivation of (1.46) we were able to carry out analytically the integral corresponding to the r-integral in (1.47).

The two results (1.46) and (1.47) are not analytically identical. This is due to the fact that the phase space integration in (1.47) includes the paths of all particles having any part of their orbits inside R, while in (1.46) we include only those particles with orbits which lie entirely within the cylinder. However, to the extent that the plasma density at the cylinder wall is negligible, the two methods should give essentially the same numerical result.

To verify this, a numerical calculation of a particular case of the lower hybrid drift instability was carried out and the results compared to those Davidson obtains from (1.47). The equilibrium distribution function is taken to be a Gibbs distribution:

$$f_{oj} = \hat{n}_j \left(\frac{m_j}{2\pi T_j}\right)^{3/2} e^{-\frac{H_\perp - \omega_j P_\theta}{T_j}} e^{-\frac{m_j v_z^2}{2T_j}} .$$

The density profile may then be shown to have the Gaussian form

$$n_{oj}(r) = \hat{n}_j \, e^{-r^2/R_o^2} , \qquad (1.48)$$

where R_o represents the characteristic radius of the plasma column and is given by

$$R_o^2 = \frac{2(ZT_e + T_i)}{Zm_e(\omega_e \omega_{ce} - \omega_e^2) - m_i(\omega_i^2 + \omega_i \omega_{ci})} . \qquad (1.49)$$

Here Z is the multiplicity of the ion charge and ω_{ce} and ω_{ci} are the absolute values of the electron and ion gyrofrequencies, respectively.

For the numerical calculation, we take k = 0, $R_c/R_o = 2.5$, $T_e/T_i = 1$, $\omega_E R_o/v_i = 3$, and $m_i m_e = 1836$, where Davidson defines ω_E as

$$R_o^2 = \frac{2(ZT_e + T_i)}{Zm_e(\omega_e \omega_{ce} - \omega_e^2) - m_i(\omega_i^2 + \omega_i \omega_{ci})} \; . \qquad (1.50)$$

Davidson carries out his calculation using the strongly magnetized electron and unmagnetized ion approximation, and the further assumption that $\lambda_n R_o > 1$, all of which are valid for the equilibrium configuration and modes dealt with in this calculation. In addition, Davidson shows that under these assumptions it is only necessary to consider the diagonal elements of the dispersion matrix to determine the unstable modes; to facilitate comparison of results we likewise use only the diagonal terms of (1.46).

The results of the calculation are shown in Figs. (1.4) - (1.8) for several n's and ℓ's. The circles represent the results obtained from (1.46), the squares Davidson's results. The two approaches are seen to be in quite good agreement. The discrepancies tend to be largest for large ℓ and small n; that is to say, for those modes where the potential fluctuations are localized near the conducting wall. As remarked above, the dispersion relation (1.46) does not include the effects of orbits extending outside the wall, so that strictly speaking the distribution function is no longer of the rigid rotor form. This does not affect the validity of (1.46) however, since the approach used to derive it is valid for any distribution function. The phase space integral leading to Davidson's result (1.47) however, includes orbits extending outside the cylinder, so that these particles are not treated correctly in the orbit integration. Davidson justifies this with the assumption that the plasma density at the wall is "negligible", due to the Gaussian form of the density function (1.48). As an example, if we take the density at the center of the cylinder to be $\hat{n}$, the density at the wall will be $n_o = e^{-6.25}\hat{n} = 1.93 \times 10^{-3}\hat{n}$, so that the plasma density at r = R is indeed negligible compared to the density at the center. However, if we consider the mode $\ell = 30$, n = 1, the maximum of $\phi_1(r)$ (and thus of $n_1(r)$) occurs at approximately $r = r_m = .90R$, so that

$n_o(r_m) \cong 3.65 \times 10^{-3}\hat{n}$. Thus in this case the density at the maximum of ϕ_1 is less than twice that at the cylinder wall, and could have a significant effect on the frequency and stability of the mode. The same may be said for other high ℓ or low n modes, and this may account for much of the difference between the two sets of results as presented in Figs. (1.4) - (1.8). We note also that Davidson finds the highest growth rate to be associated with the $\ell = 44$, $n = 1$ mode, which is localized close enough to the wall to be affected by the above considerations. In fact, the difference in the two results for this mode, as shown in Fig. (1.8), amounts to about 10%, so that the effect of finite density at the wall seems to be non-negligible for the most unstable modes.

Fig 1.1

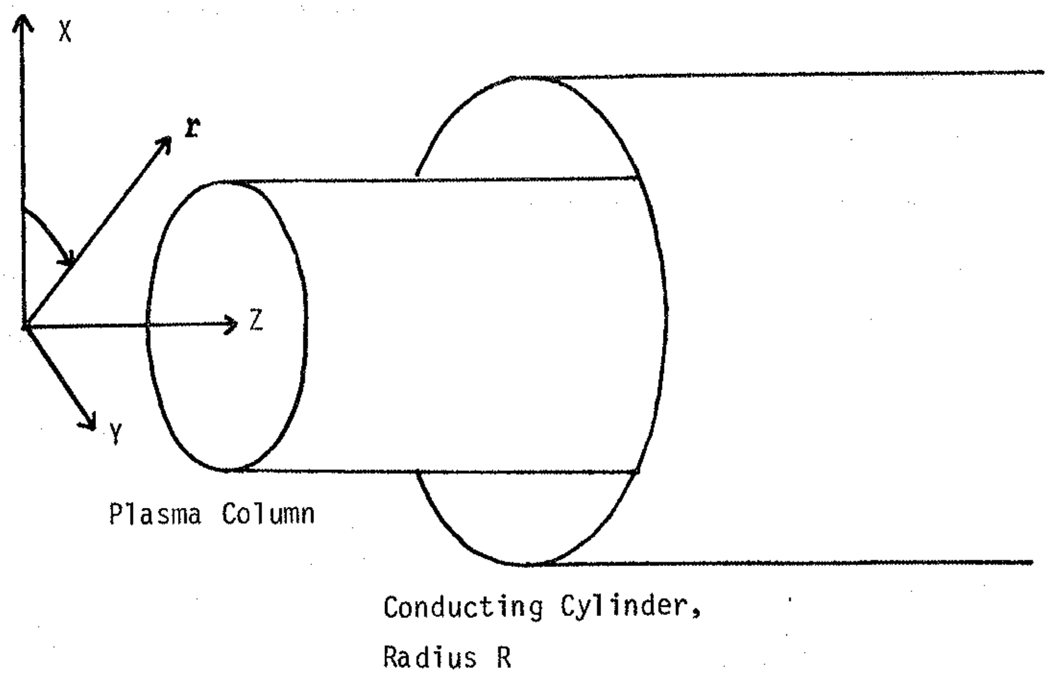
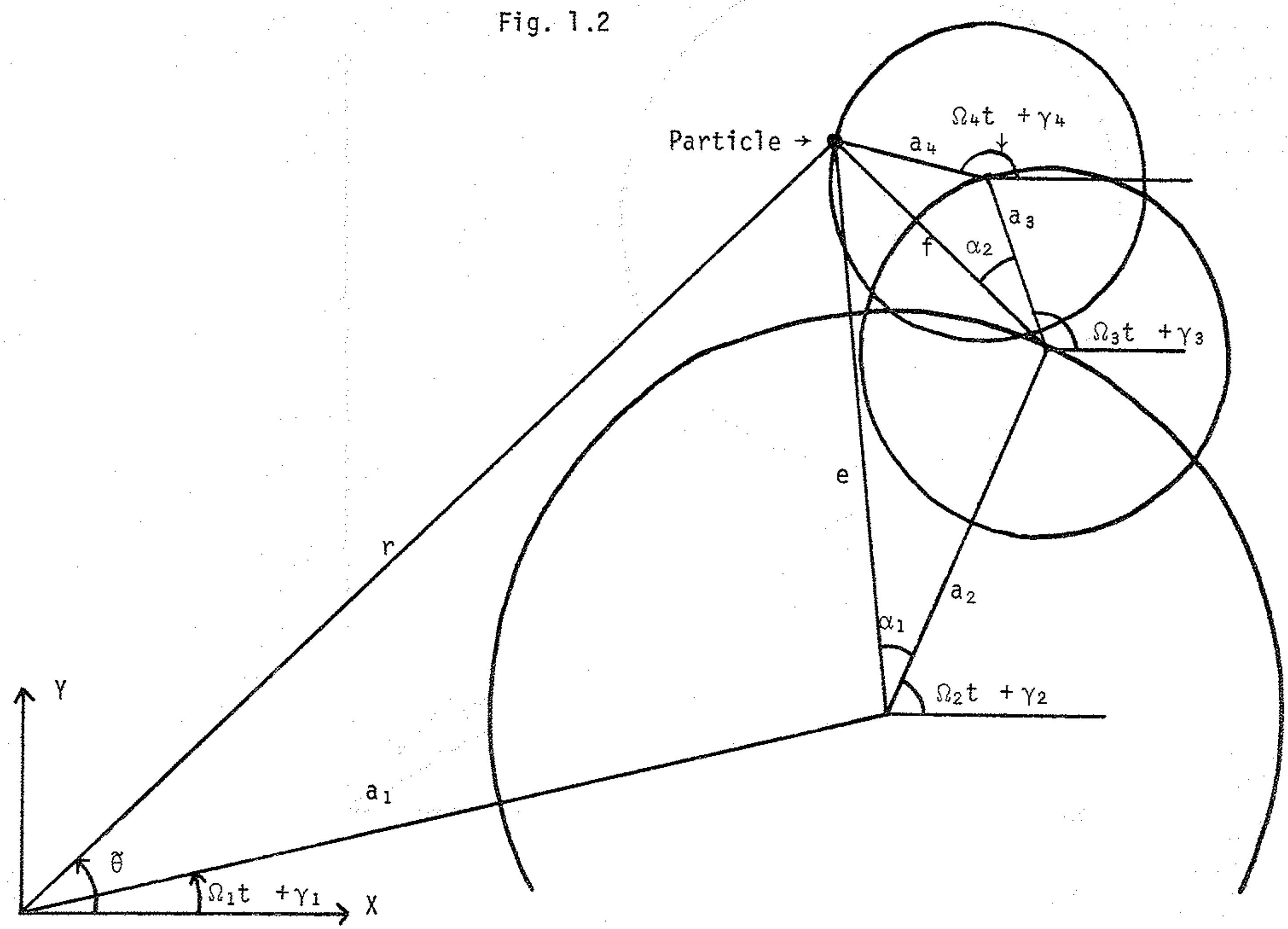


Fig. 1.2


Particle →
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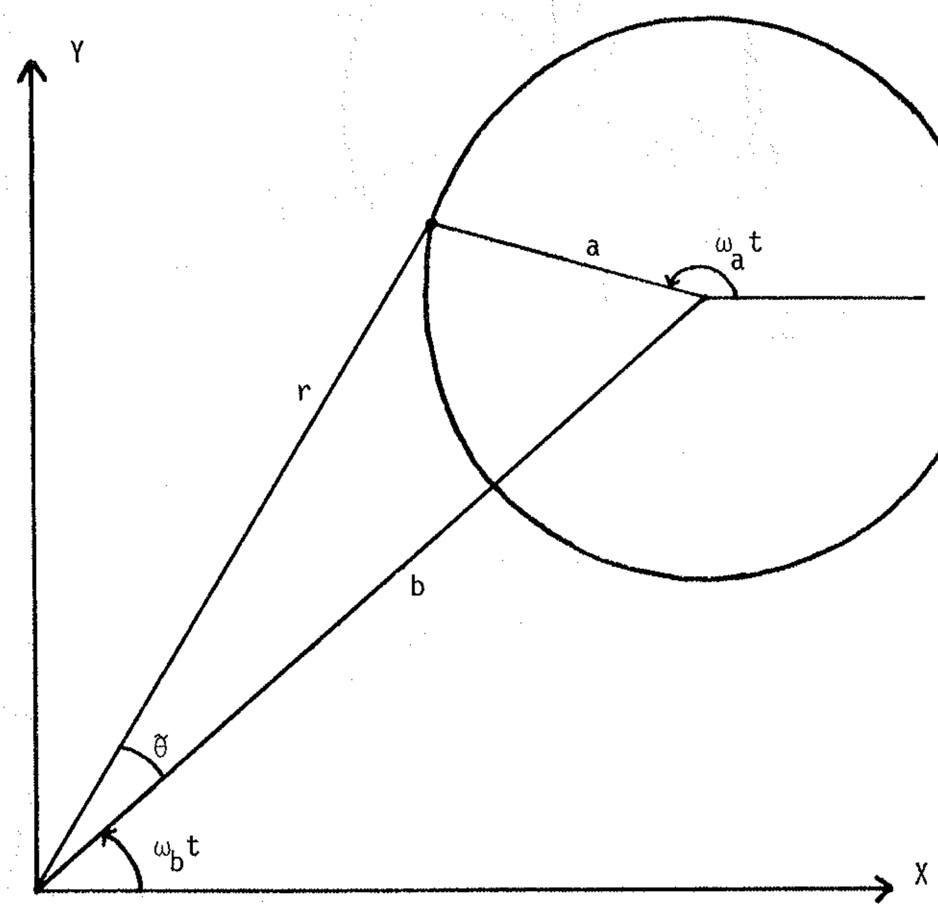
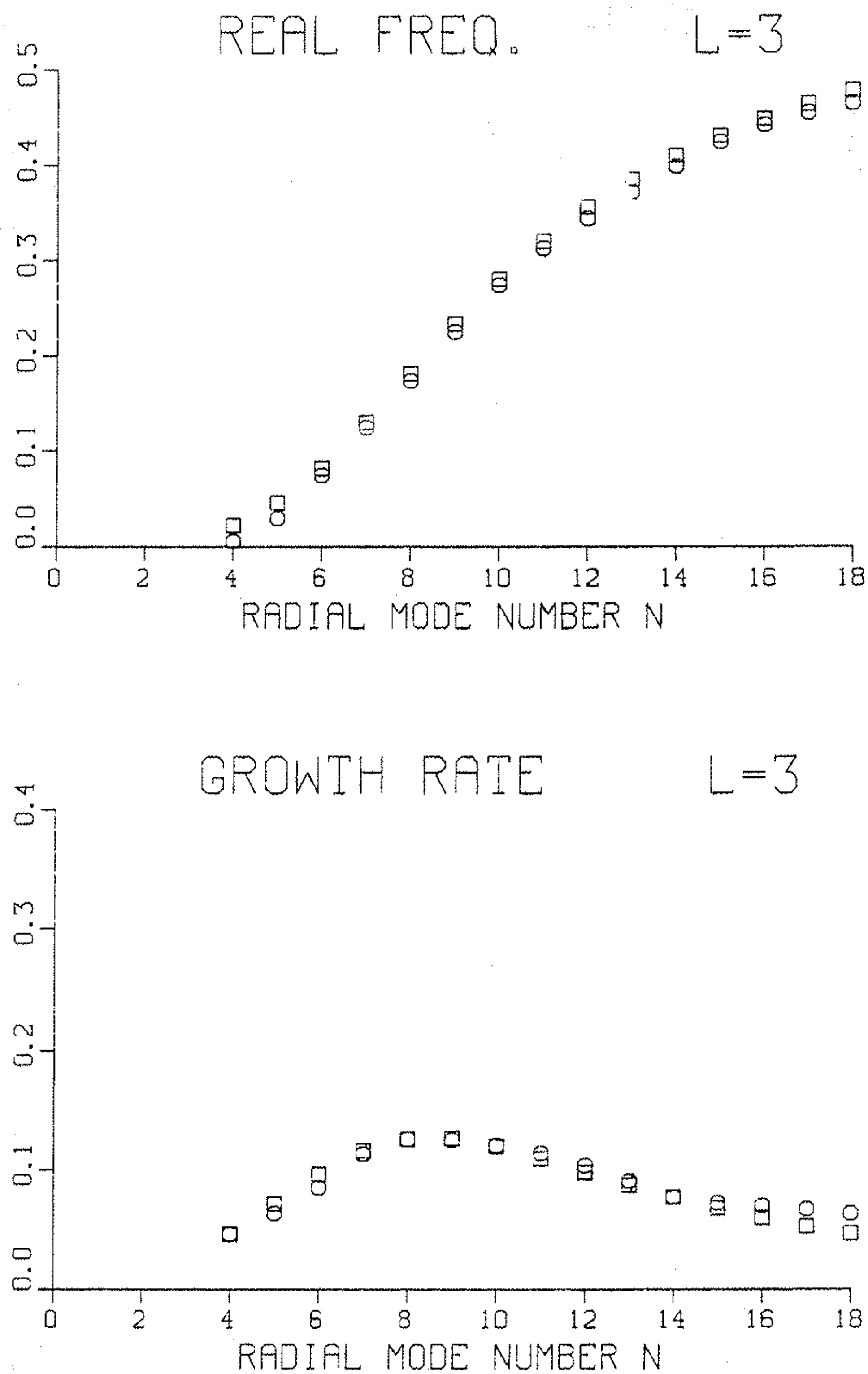
a_4
a_3
α_2
f
$\Omega_3 t + \gamma_3$
e
r
a_2
α_1
$\Omega_2 t + \gamma_2$
Y
a_1
θ
$\Omega_1 t + \gamma_1$
X

Fig. 1.3

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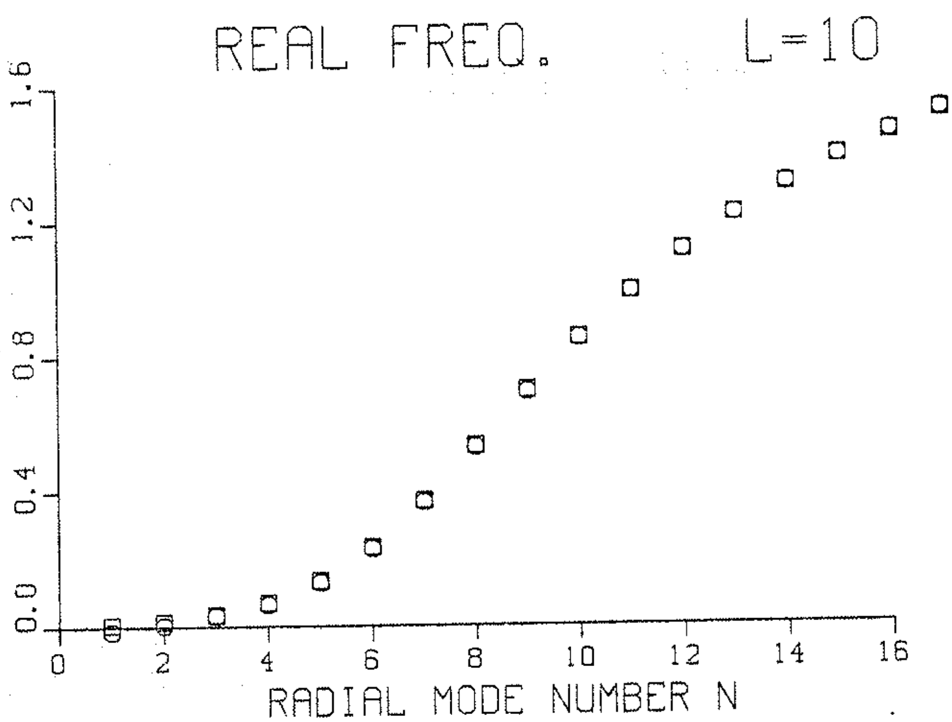
0.5
0
2
4
6
8
10
12
14
16
18
RADIAL MODE NUMBER N
GROWTH RATE
L=3
0.0
0.1
0.2
0.3
0.4
0
2
4
6
8
10
12
14
16
18
RADIAL MODE NUMBER N

Fig. 1.4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1.6
0
2
4
6
8
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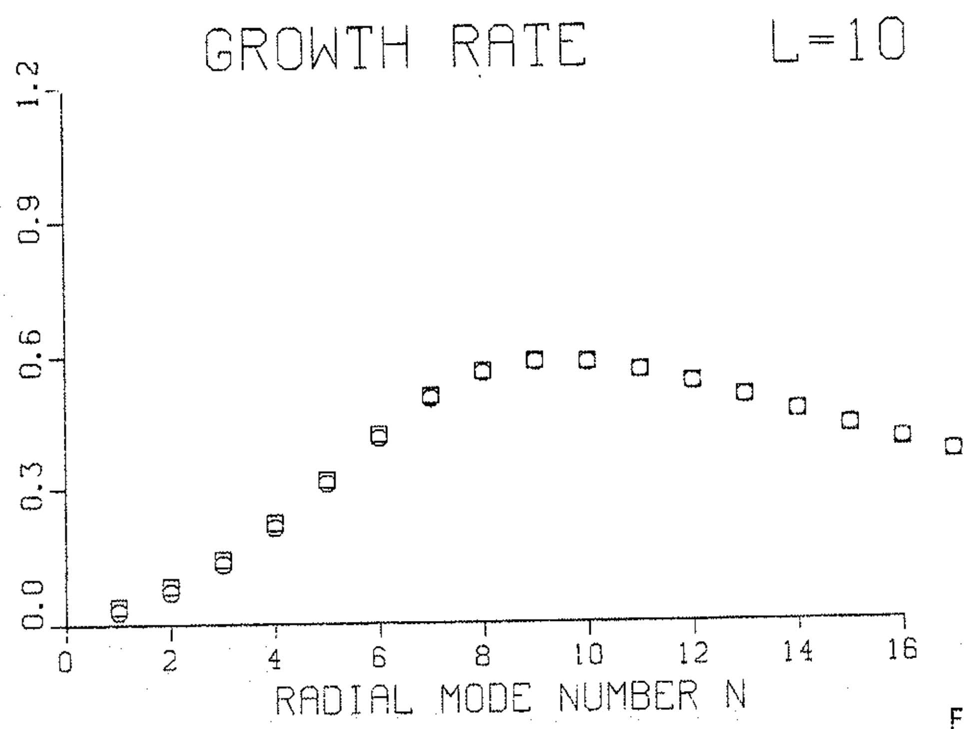
12
14
16
RADIAL MODE NUMBER N


GROWTH RATE
L=10
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0.3
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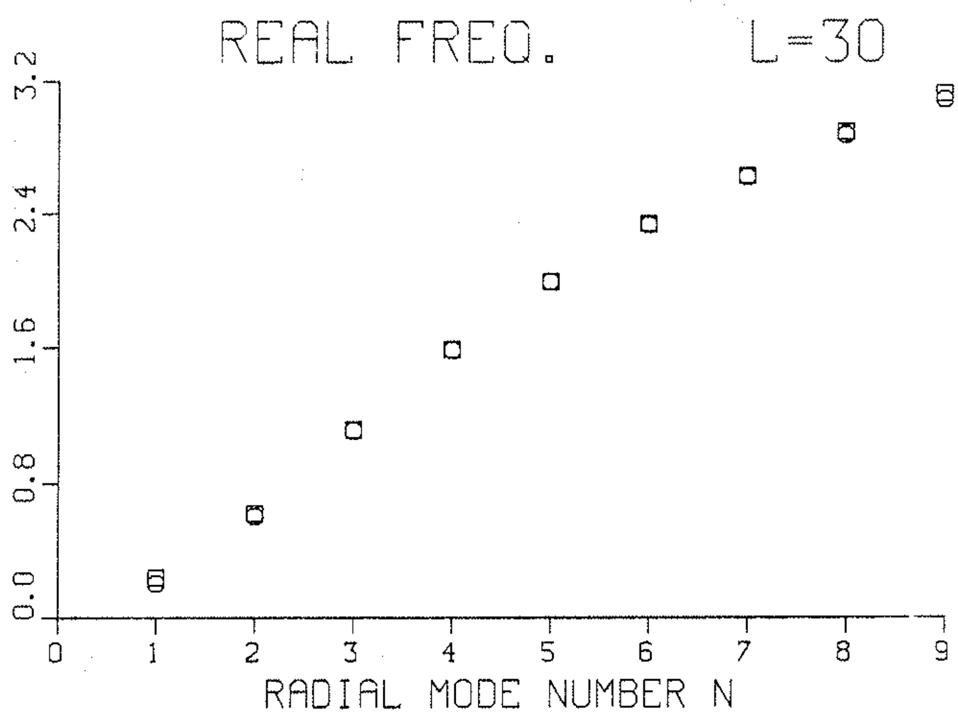
1.2
0
2
4
6
8
10
12
14
16
RADIAL MODE NUMBER N

Fig 1.5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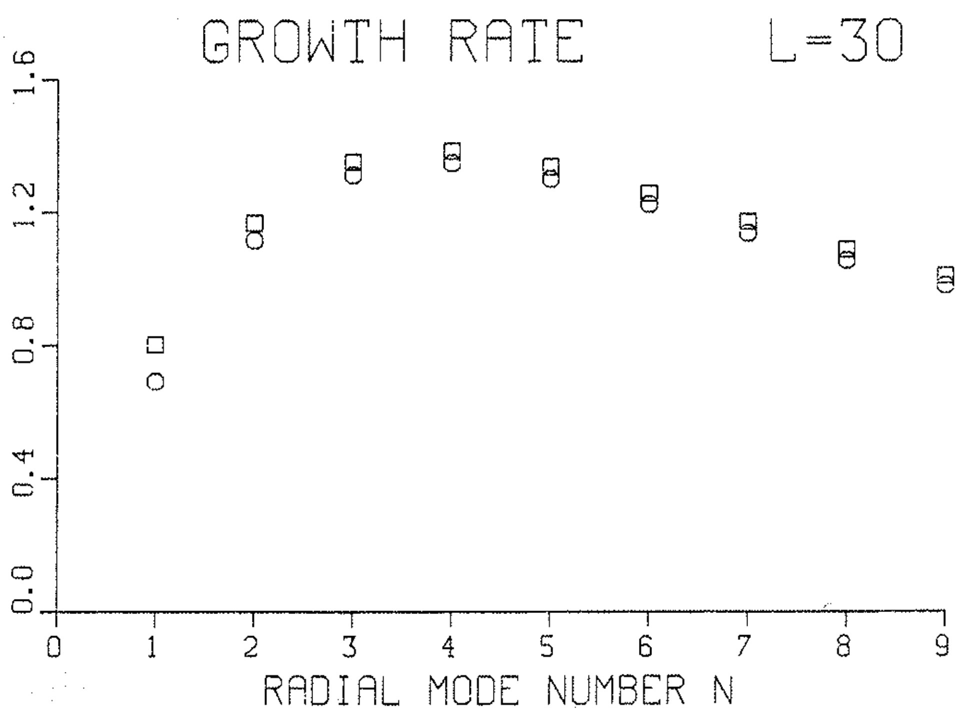
1.6
2.4
3.2
0 1 2 3 4 5 6 7 8 9
RADIAL MODE NUMBER N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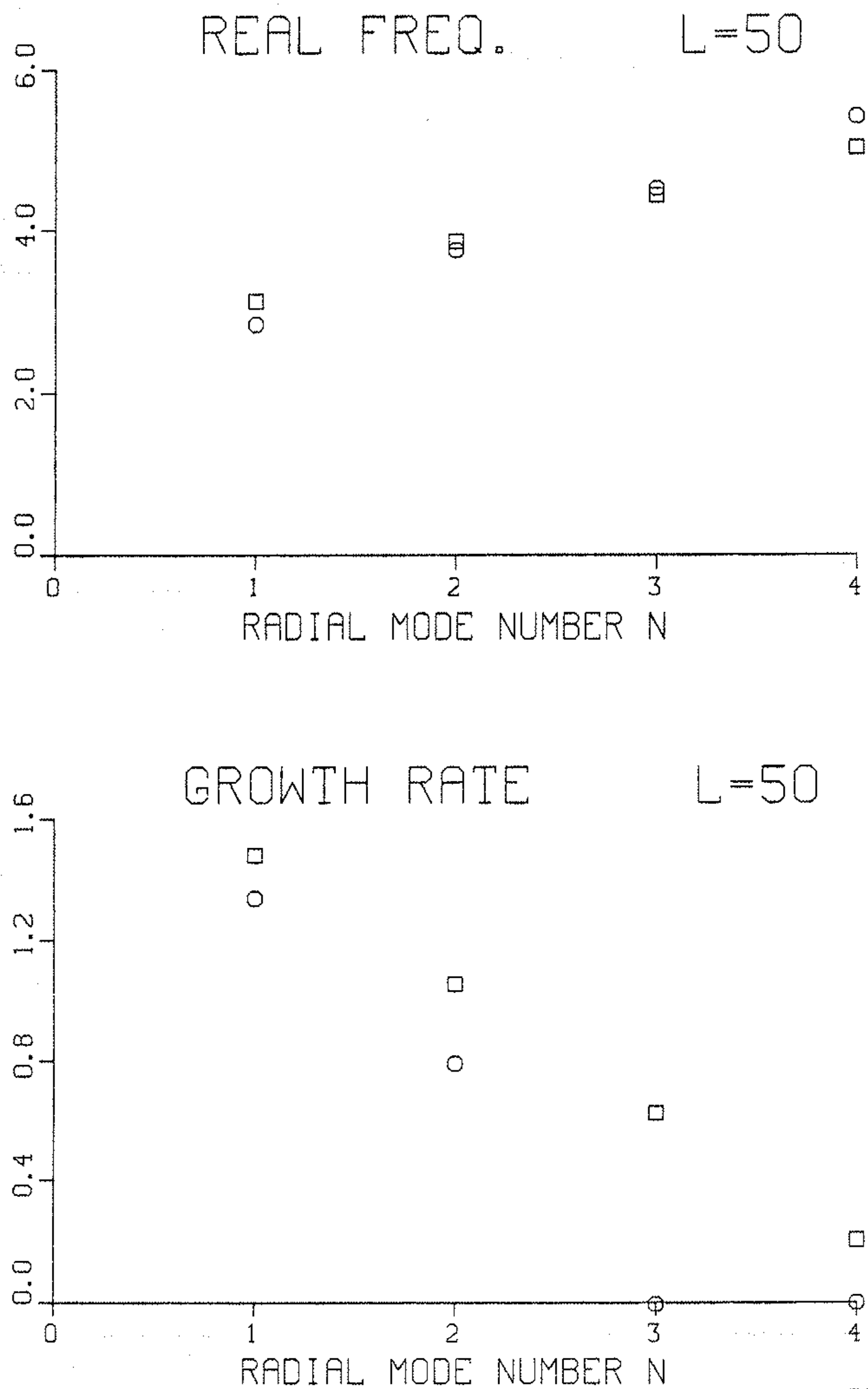
1.2
1.6
0 1 2 3 4 5 6 7 8 9
RADIAL MODE NUMBER N

Fig 1.6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4.0
2.0
0.0
0
1
2
3
4
RADIAL MODE NUMBER N
GROWTH RATE   L=50
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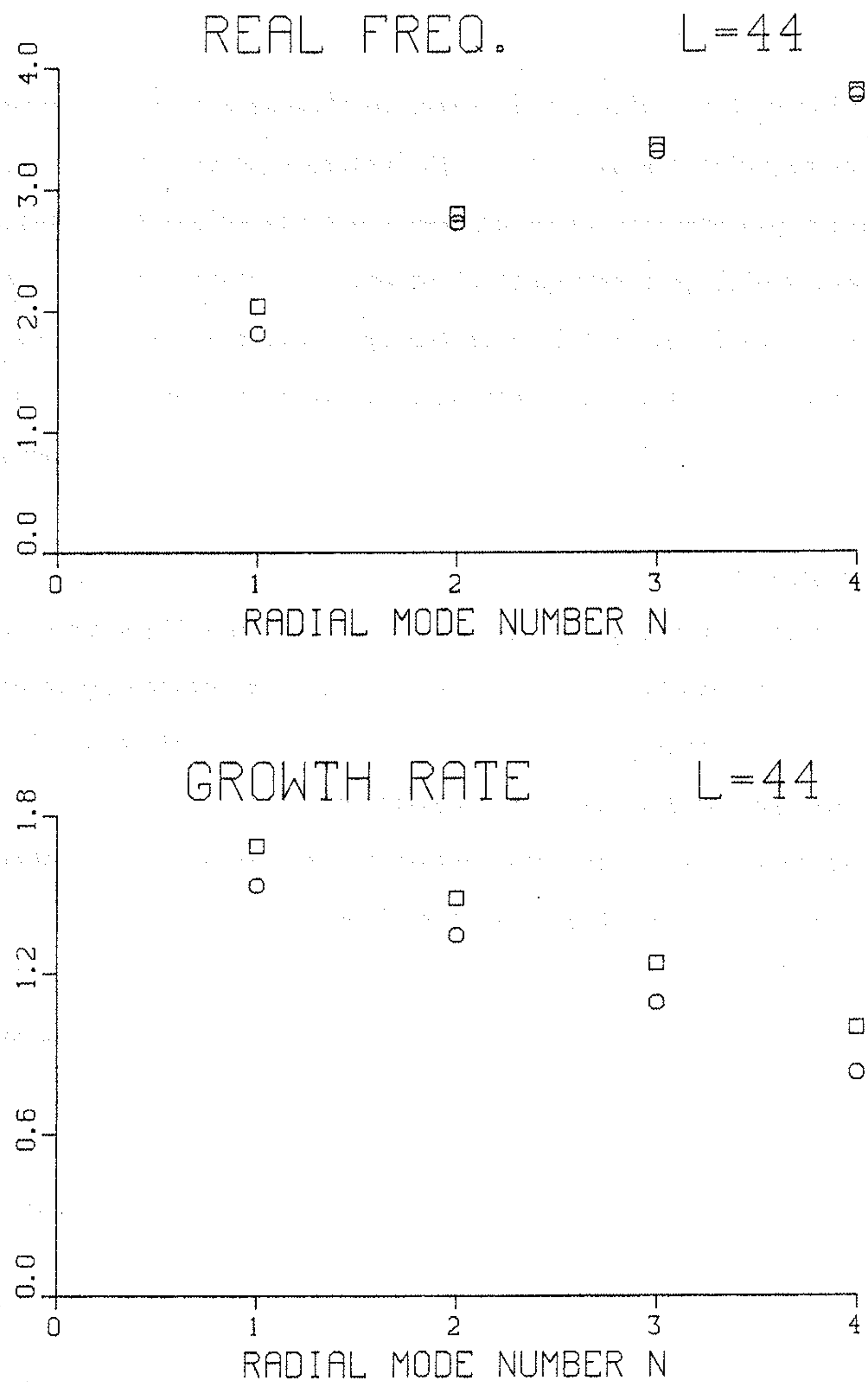
1.2
0.8
0.4
0.0
0
1
2
3
4
RADIAL MODE NUMBER N

Fig. 1.7


REAL FREQ. L=44
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3.0
2.0
1.0
0.0
0
1
2
3
4
RADIAL MODE NUMBER N
GROWTH RATE L=44
1.8
1.2
0.6
0.0
0
1
2
3
4
RADIAL MODE NUMBER N

Fig. 1.8

II. EXTENSION TO ELECTROMAGNETIC CASE

For problems with $\beta \sim 1$, where considerable free energy resides in the magnetic field, or for high frequency modes with $\omega \sim ck$, we expect significant coupling between the longitudinal (electrostatic) and transverse (electromagnetic) modes. In relativistic plasmas, where a significant fraction of the particles have $v/c \cong 1$, we expect that fundamentally electromagnetic modes may be driven unstable by resonant particles. Consequently, for many problems an accurate stability analysis will require consideration of transverse as well as longitudinal modes.

In the preceding chapter we have shown how a linearized stability analysis may be carried out for the Vlasov-Poisson equations in cylindrical geometry. In this chapter we extend this analysis to the full set of Vlasov-Maxwell equations and in the following chapter we shall illustrate this formalism with two numerical analyses of relativistic beam-plasma interactions.

We again assume the geometry and coordinate system of Fig. (1.1). The linearized Vlasov-Maxwell equations in the Lorentz gauge are:

$$\left(\frac{\partial}{\partial t} + L_o\right) f_{1j}(\underline{r},\underline{v},t) =$$

$$\frac{e_j}{m_j}\left[\nabla\phi_1(\underline{r},t) - \frac{1}{c}\,\underline{v}\times(\nabla\times\underline{A}_1(\underline{r},t)) + \frac{1}{c}\frac{\partial}{\partial t}\underline{A}_1(\underline{r},t)\right]\cdot\frac{\partial f_{oj}}{\partial \underline{v}} \quad , \tag{2.1}$$

$$(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})\, \underline{A}_1(\underline{r},t) = -\sum_j \frac{4\pi e_j}{c}\int d^3v\underline{v}f_{1j}(\underline{r},\underline{v},t) \;,$$

$$(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})\, \phi_1(\underline{r},t) = -\sum_j 4\pi e_j\int d^3v f_{1j}(\underline{r},\underline{v},t) \;, \qquad (2.2)$$

with gauge condition

$$\nabla \cdot \underline{A}_1 + \frac{1}{c}\frac{\partial \phi_1}{\partial t} = 0 \;. \qquad (2.3)$$

We assume that all perturbed quantities have $e^{-i\omega t}$ time dependence, with $\mathrm{Im}(\omega) > 0$.

The Lorentz condition (2.3) does not uniquely specify $\underline{A}$, as we may introduce a further restricted gauge transformation[8]

$$\underline{A} \rightarrow \underline{A} + \nabla\Lambda \;,$$

$$\phi \rightarrow \phi - \frac{1}{c}\frac{\partial \Lambda}{\partial t} = \phi + \frac{i\omega}{c}\Lambda \;, \qquad (2.4)$$

where $\Lambda(r,\theta,z,t)$ is any function satisfying

$$(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2})\Lambda = (\nabla^2 + \frac{\omega^2}{c^2})\Lambda = 0 \;. \qquad (2.5)$$

We now show that we can use the transformation (2.4) to pick a gauge in which the potentials satisfy the boundary conditions

$$\nabla \cdot \underline{A}_1\Big|_{r=R} = \phi_1\Big|_{r=R} = 0 \;, \qquad (2.6)$$

where R is again the radius of the conducting cylinder.

Since the potentials are periodic in the θ and z coordinates, we may write

$$\phi(R,\theta,z) = \sum_{\ell,k} \Phi(\ell,k)\ e^{i(\ell\theta+kz)}\ . \qquad (2.7)$$

If we define $\Lambda(r,\theta,z)$ by

$$\Lambda(r,\theta,z) = \sum_{\ell,k} \Lambda(\ell,k)\ J_\ell(\lambda_{\ell,k}r)\ e^{i(\ell\theta+kz)}\ , \qquad (2.8)$$

where $\Lambda(\ell,k)$ and $\lambda_{\ell,k}$ are constants depending on ℓ and k, then we have

$$(\nabla^2 + \frac{\omega^2}{c^2})\Lambda = \sum_{\ell,k} (\frac{\omega^2}{c^2} - \lambda_{\ell,k}^2 - k^2)\ J_\ell(\lambda_{\ell,k}r)\ e^{i(\ell\theta+kz)}\ . \qquad (2.9)$$

Clearly Λ will satisfy (2.9) if we choose

$$\lambda_{\ell,k} = \sqrt{\frac{\omega^2}{c^2} - k^2}\ , \qquad (2.10)$$

and from (2.7) and (2.8) we see that (2.6) will be satisfied if

$$\Lambda(\ell,k) = \Phi(\ell,k)/\ \frac{i\omega}{c}\ J_\ell(\lambda_{\ell,k}R)\ . \qquad (2.11)$$

Note that since $\mathrm{Im}(\omega) > 0$, (2.10) shows that $\lambda_{\ell,k}$ is always complex (non-real), and since all the zeros of the Bessel functions of the first kind are real, the denominator in (2.11) never vanishes. Consequently, we may always impose the boundary condition (2.6), and in doing so uniquely determine the gauge of $\underline{A}$ and ϕ (up to an additive constant in $\underline{A}$).

Using the gauge condition (2.5) we may eliminate ϕ from the equations:

$$\phi = -\frac{ic}{\omega} \nabla \cdot \underline{A} \, , \tag{2.12}$$

where we now drop the subscript "ℓ" on the perturbed quantities. E_θ, E_z, and B_r must be zero at the cylinder wall, so from (2.6), (2.12), and

$$\underline{E} = -\nabla\phi + \frac{i\omega}{c} \underline{A}$$

we have for boundary conditions on $\underline{A}$:

$$\nabla \cdot \underline{A}\Big|_{r=R} = 0 \quad , \quad A_\theta\Big|_{r=R} = 0 \quad , \quad A_z\Big|_{r=R} = 0 \, . \tag{2.13}$$

In the preceding chapter, we expanded the scalar potential in cylindrical harmonics in order to obtain a matrix dispersion relation. These expansion functions were chosen to satisfy the boundary condition $\phi(r) = 0$. For the electromagnetic case we now define

$$A^{\pm} = A_x \pm iA_y \, ,$$

and show that a suitable expansion for the vector potential satisfying the boundary conditions (2.13) is the following:

$$A^+(\underline{r}) = \sum_{\ell,n,k} [\alpha_{n\ell k}\, J_{\ell+1}(\lambda'_{n\ell}r) + \beta_{n\ell k}\, J_{\ell+1}(\lambda_{n\ell}r)]\, e^{i[(\ell+1)\theta+kz]} ,$$

$$A^-(\underline{r}) = \sum_{\ell,n,k} [\alpha_{n\ell k}\, J_{\ell-1}(\lambda'_{n\ell}r) - \beta_{n\ell k}\, J_{\ell-1}(\lambda_{n\ell}r)]\, e^{i[(\ell-1)\theta+kz]} ,$$

$$A_z(\underline{r}) = \sum_{\ell,n,k} \gamma_{n\ell k}\, J_\ell(\lambda_{n\ell}r)\, e^{i(\ell\theta+kz)} . \qquad (2.14)$$

Here $\alpha_{n\ell k}$, $\beta_{n\ell k}$, $\gamma_{n\ell k}$ are the expansion coefficients representing the three independent components of the potentials, $\lambda_{n\ell}$ is the n^{th} root of $J_\ell(R) = 0$ and $\lambda'_{n\ell}$ is the n^{th} root of $J_\ell(R) = 0$. Thus the third equation in (2.14) shows that the third boundary condition in (2.13) is satisfied. To show that the second boundary condition is satisfied, we must express A_θ in terms of A^+ and A^-. We have

$$A_\theta = -A_x \sin\theta + A_y \cos\theta \quad , \quad A_x = \frac{1}{2}(A^+ + A^-) \quad , \quad A_y = \frac{1}{2i}(A^+ - A^-) ,$$

and thus

$$A_\theta = -\frac{1}{2}(A^+ + A^-)\sin\theta - \frac{i}{2}(A^+ - A^-)\cos\theta = \frac{i}{2}[A^- e^{i\theta} - A^+ e^{-i\theta}] .$$

Using (2.14) this gives

$$A_\theta = \frac{i}{2} \sum_{\ell,n,k} \{\alpha_{n\ell k}\, [J_{\ell=1}(\lambda'_{n\ell}r) - J_{\ell+1}(\lambda'_{n\ell}r)]$$

$$- \beta_{n\ell k}\, [J_{\ell-1}(\lambda_{n\ell}r) + J_{\ell+1}(\lambda_{n\ell}r)]\}\, e^{i(\ell\theta+kz)} .$$

Using the Bessel function identities

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x} J_n(x) \quad , \quad J_{n-1}(x) - J_{n+1}(x) = 2J_n'(x)$$

we then have

$$A_\theta = i \sum_{\ell,n,k} \{\alpha_{n\ell k} [J_\ell'(\lambda_{n\ell}' r) - \beta_{n\ell k} \cdot \frac{\ell}{\lambda_{n\ell} r} J_\ell(\lambda_{n\ell} r)] \, e^{i(\ell\theta+kz)}\} \ .$$

Since

$$J_\ell(\lambda_{n\ell} r) = J_\ell'(\lambda_{n\ell}' R) = 0 \ ,$$

we see that the second of the boundary conditions (2.13) is satisfied.

From a similar though longer calculation (given in Appendix B), we find:

$$\nabla \cdot \underline{A} = \sum_{\ell,n,k} (\beta_{n\ell k} \lambda_{n\ell} + ik_z \gamma_{n\ell k}) J_\ell(\lambda_{n\ell} r) \, e^{i(\ell\theta+kz)}, \qquad (2.15)$$

and thus the first of (2.13) is also satisfied.

As in the electrostatic case, we now determine the perturbed distribution functions by integration over unperturbed orbits. From (2.1) we obtain:

$$f_{1j}(\underline{r},\underline{v}) = \frac{e_j}{m_j} \int_{-\infty}^{t} dt' \, e^{i\omega(t-t')} [\nabla\phi(\underline{r}) - \frac{i\omega}{c} \underline{A}(r) - \frac{\underline{v}\times(\nabla\times\underline{A}(\underline{r}))}{c}] \cdot \frac{\partial f_{oj}}{\partial \underline{v}} \ . \qquad (2.16)$$

Using

$$\frac{\partial f_{oj}}{\partial \underline{v}} = \frac{\partial f_{oj}}{\partial H} m_j \underline{v} + \frac{\partial f_{oj}}{\partial P_\theta} m_j r\theta + \frac{\partial f_{oj}}{\partial P_z} m_j \hat{z} \tag{2.17}$$

and substituting (2.14) into (2.16), we obtain after considerable algebra:

$$f_{1j} = e_j \int_{-\infty}^{t} dt'\, e^{i\omega(t-t')} \sum_{\ell,n,k_z} \Bigg\{ \frac{\partial f_{oj}}{\partial H} \left(-\frac{ic}{\omega}\right)\Big[(\lambda_{n\ell}\beta_{n\ell k} + ik_z\gamma_{n\ell k}) \frac{d}{dt'} C_{n\ell}$$

$$+ \frac{1}{2}\alpha_{n\ell k}(C'_{n\ell+1}v'_- + C'_{n\ell-1}v'_+) + \frac{1}{2}\beta_{n\ell k}(C_{n\ell+1}v'_- - C_{n\ell-1}v'_+)\Big]$$

$$+ \frac{\partial f_{oj}}{\partial P_\theta}\Big[\frac{\ell c}{\omega}(\lambda_{n\ell}\beta_{n\ell k} + ik_z\gamma_{n\ell k})\,\overline{C}_{n\ell} - \frac{i\ell v'_z}{c}\gamma_{n\ell k}\,\overline{C}_{n\ell}$$

$$- \frac{1}{2}\frac{k_z v'_z}{c}\alpha_{n\ell k}(r'_+ C'_{n\ell-1} - r'_- C'_{n\ell+1}) + \frac{\ell k_z v'_z}{\lambda_{n\ell} c}\beta_{n\ell k}\overline{C}_{n\ell}$$

$$+ \frac{\omega\alpha_{n\ell k}}{2c}(C'_{n\ell-1}r'_+ - C'_{n\ell+1}r'_-) \tag{2.18}$$

$$- \frac{\omega\beta_{n\ell k}}{2c}(C_{n\ell-1}r'_+ + C_{n\ell+1}r'_-) - \frac{i\lambda'_{n\ell}}{2c}(v'_+r'_- + v'_-r'_+)\,\alpha_{n\ell k}\,\overline{C}'_{n\ell}$$

$$+ \frac{\partial f_{oj}}{\partial P_z}\Big[\frac{k_z c}{\omega}(\lambda_{n\ell}\beta_{n\ell k} + ik_z\gamma_{n\ell k})\,\overline{C}_{n\ell} - \frac{\lambda_{n\ell}}{2c}\gamma_{n\ell k}$$

$$\cdot (v'_-C_{n\ell+1} - v'_+C_{n\ell-1}) + \gamma_{n\ell k}\,\overline{C}_{n\ell}$$

$$- \frac{ik_z}{2c}\alpha_{n\ell k}(v'_+C'_{n\ell-1} + v'_-C'_{n\ell+1}) - \frac{ik_z}{2c}\beta_{n\ell k}(v'_-C_{n\ell+1} - v'_+C_{n\ell-1})\Big]\Bigg\} .$$

In (2.18)

$$v'_{\pm} = v'_x(t') \pm i v'_y(t') \quad , \quad r'_{\pm} = x'(t') \pm i y'(t') \; .$$

Here the prime indicates dependence on t' and

$$\underline{r}'(t'=t) = \underline{r} \quad , \quad \underline{v}'(t'=t) = \underline{v} \; ,$$

and we have defined

$$\overline{C}_{n\ell} = J_\ell(\lambda_{n\ell} r') e^{i(\ell\theta' + k_z z')} \; , \; C_{n\ell\pm1} = J_{\ell\pm1}(\lambda_{n\ell} r') e^{i[(\ell\pm1)\theta' + k_z z']} \; ,$$

$$\overline{C}'_{n\ell} = J_\ell(\lambda'_{n\ell} r') e^{i(\ell\theta' + k_z z')} \; , \; C'_{n\ell\pm1} = J_{\ell\pm1}(\lambda'_{n\ell} r') e^{i[(\ell\pm1)\theta' + k_z z']} \; .$$

Now we can use (1.24), (1.25) and (1.28) to express $e^{ik_z z'}$, $r'_{\pm}$, $v'_{\pm}$, and the C's as Fourier series in time and proceed with the integration as in the electrostatic case, obtaining a set of linear equations in the expansion coefficients $\alpha_{n\ell k}$, $\beta_{n\ell k}$, $\gamma_{n\ell k}$. This is quite cumbersome for the general equation (2.18), however, so for the purpose of illustration and with a view toward the applications in the next chapter we carry out the calculation in full only for the simple case of $k = A_z = 0$, i.e., we consider only perturbations depending solely on r and θ. We note, however, that the extension to z-dependent perturbations, though algebraically lengthy, is straightforward.

We therefore take $\gamma_{n\ell k} = k = 0$, so that Eq. (2.18) becomes

$$f_{1j} = e_j \int_{-\infty}^{t} dt' \, e^{i\omega(t-t')} \sum_{\ell,n} \Bigg[\frac{\partial f_{oj}}{\partial H} \Big\{ - \frac{ic}{\omega} \beta_{n\ell}\lambda_{n\ell} \frac{d}{dt'} [J_\ell(\lambda_{n\ell} r') e^{i\ell\theta'}]$$

$$- \frac{i\omega}{2c} \alpha_{n\ell} [J_{\ell+1}(\lambda'_{n\ell} r') v'_- \, e^{i(\ell+1)\theta'} + J_{\ell-1}(\lambda'_{n\ell} r') \, v'_+ \, e^{i(\ell-1)\theta'}]$$

$$- \frac{i\omega}{2c} \lambda_{n\ell} [J_{\ell+1}(\lambda_{n\ell} r') v'_- \, e^{i(\ell+1)\theta'} - J_{\ell-1}(\lambda_{n\ell} r') v'_+ \, e^{i(\ell-1)\theta'}] \Big\}$$

$$+ \frac{\partial f_{oj}}{\partial P_\theta} \Big\{ \frac{\ell c}{\omega} \lambda_{n\ell} \lambda_{n\ell} \, J_\ell(\lambda_{n\ell} r') e^{i\ell\theta'} \tag{2.19}$$

$$- \frac{i}{2c} (v'_+ r'_- + v'_- r'_+) \, \alpha_{n\ell} \lambda'_{n\ell} \, J_\ell(\lambda'_{n\ell} r') e^{i\ell\theta'} + \frac{\omega}{2c} \alpha_{n\ell}$$

$$\cdot \, [J_{\ell-1}(\lambda'_{n\ell} r') r'_+ \, e^{i(\ell-1)\theta'} - J_{\ell+1}(\lambda'_n \, r') r'_- e^{i(\ell+1)\theta'}]$$

$$- \frac{\omega}{2c} \beta_{n\ell} [J_{\ell-1}(\lambda_n \, r') r'_+ \, e^{i(\ell-1)\theta'} + J_{\ell+1}(\lambda_{n\ell} r') r'_- \, e^{i(\ell+1)\theta'}] \Big\} \Bigg] \; .$$

Next we use the techniques of the previous chapter to represent the integrand of (2.19) as a Fourier series in time. Since we still have only one non-ignorable coordinate, the arguments which in the electrostatic case led to the expansions (1.25) and (1.28) remain valid, and terms of the form $r'_\pm$, $v'_\pm$, and $J_\ell(r')e^{i\ell\theta'}$ may all be readily represented as Fourier series in time. To make the method as clear as possible and avoid unduly lengthy algebra, we utilize only the first two terms in the expansions (1.25) and (1.28) in the following calculations. (The extension to more than two terms is again straightforward, as in the electrostatic case.) Two terms