

STIMULATED THOMSON SCATTERING

BY

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A thesis submitted in partial fulfillment of the  
requirements for the degree of

DOCTOR OF PHILOSOPHY

(Physics)

at the

UNIVERSITY OF WISCONSIN - MADISON

1979

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To Margo

## ACKNOWLEDGEMENTS

I thank L. W. Anderson and Jim Lawler for bringing this problem to my attention, and for many fruitful discussions. I also thank my adviser, K. R. Symon, for his careful reading of this manuscript and for many valuable discussions. I am grateful to the Danforth Foundation, the Plasma Physics Group, and the Physics Department for their support of my graduate study. Especially I thank my wife, Margo, and my children, Melanie, David, and Reed, for their patience, love, and support. I thank also my parents, Loren and LaRene Spencer, for their unfailing love and encouragement.

This research has been supported by the Danforth Foundation, by The Department of Energy, and by the Wisconsin Alumni Research Foundation, Madison, Wisconsin.

## TABLE OF CONTENTS

ACKNOWLEDGEMENTS	iv
LIST OF FIGURES	vii
LIST OF SYMBOLS	ix
ABSTRACT	xi
CHAPTER 1. INTRODUCTION	1
CHAPTER 2. QUANTUM MECHANICAL THEORY OF STIMULATED THOMSON SCATTERING	9
CHAPTER 3. FLUID THEORY OF STIMULATED THOMSON SCATTERING	
A. Introduction	17
B. A Perturbation Expansion of the Electron Fluid Equations and Maxwell's Equations	18
C. The Motion of the Electron Fluid Driven by the Ponderomotive Force for Weak Plasma Response	22
D. The Gain and Dispersion of the Waves	27
E. Comparison with the Quantum Mechanical Result; Kramers-Kronig Relations	35
F. The Gain and Dispersion for General Plasma Response	38
G. An Approximation to the Inhomogeneously	

Broadened Limit	70
CHAPTER 4. INDEPENDENT ELECTRON THEORY OF STIMULATED THOMSON SCATTERING	
A. Gain and Dispersion in the Plasma	72
B. Particle Orbits	76
C. The Gain, $\theta$ , in the Homogeneously Broadened Limit	83
D. The Phase Shift, $\phi$ , in the Homogeneously Broadened Limit	93
E. The Gain and the Phase Shift in the Inhomogeneously Broadened Limit	99
CHAPTER 5. SUMMARY	109
APPENDIX 1. THE FREE ELECTRON AMPLIFIER	112
APPENDIX 2. THE PERTURBED DENSITY AND THE FUNCTIONS J AND K FOR GENERAL $\Gamma$ AND $\omega'$	120
APPENDIX 3. PROPOSED EXPERIMENTAL DETECTION OF STIMULATED THOMSON SCATTERING	127
APPENDIX 4. PROPERTIES OF JACOBIAN ELLIPTIC FUNCTIONS	132
BIBLIOGRAPHY	134

## LIST OF FIGURES

FIGURE	TITLE	PAGE
1.	The Gain Curve for Unsaturated Stimulated Thomson Scattering	32
2.	The Dispersion Curve for Unsaturated Stimulated Thomson Scattering	34
3.	The Gain Curve for Underdamping and $\tau$ Fixed	41
4.	The Gain Curve for Critical Damping and $\tau$ Fixed	43
5.	The Gain Curve for Overdamping and $\tau$ Fixed	45
6.	The Dispersion Curve for Underdamping and $\tau$ Fixed	47
7.	The Dispersion Curve for Critical Damping and $\tau$ Fixed	49
8.	The Dispersion Curve for Overdamping and $\tau$ Fixed	51
9.	The Gain Curve for Underdamping and $\delta\omega$ Fixed	53
10.	The Gain Curve for Critical Damping and $\delta\omega$ Fixed	55
11.	The Gain Curve for Overdamping	

In 1917, Albert Einstein wrote a brilliant paper<sup>2</sup> on the connection between the Planck radiation law and the quantum mechanics of radiation. He proposed the process of stimulated emission, wherein the rate at which a system makes a radiative transition at frequency  $\omega$  is proportional to the energy density of radiation evaluated at the frequency  $\omega$ . He then showed that this assumption, together with the Boltzmann relation relating the relative numbers of states with different energies, implies a radiation law of the Planck form. The argument is very general, and suggests that any radiative transition can be stimulated.

In 1933, Kapitza and Dirac<sup>3</sup> considered the connection between Thomson's simple radiation mechanism and Einstein's stimulated emission. They calculated the effect that stimulated Thomson scattering would have on electrons shot through a pattern of standing light waves produced by coherent light reflecting onto itself from a mirror. They proposed that the effect be detected by observing the recoil of the electrons that produce the scattering. Unfortunately, the recoil is so slight that the effect is very hard to detect in this way; in fact, the experiment has never been successfully performed.

More recently, a few papers appeared which concentrated not on the effect of stimulated Thomson



scattering on the electrons, but on its effect on the amplitudes of the electromagnetic waves involved<sup>4-9</sup>. The usual arrangement proposed allows two counter-propagating electromagnetic waves of different frequencies to impinge on electrons. If an electron absorbs a photon from one of the waves and emits a photon into the propagation direction of the other wave, and if the electron's velocity is just right, the emitted photon will match in frequency and wave vector the photons in the second wave. Such electrons form a system capable of making a radiative transition that may be stimulated by the presence of the wave that receives the scattered photon. The process works, of course, in both directions, causing both stimulated emission and absorption in each wave. However, because of the electron recoil during absorption and emission, the velocity of an electron that causes energy transfer to one of the waves is different from the velocity that causes the inverse process to take place. If there are more electrons with one velocity than the other, there will be net energy transfer from one wave to the other. The possibility of detecting this energy transfer was at first considered to be very small because of the large coherent electromagnetic fields required. Present lasers may be capable of overcoming this difficulty, but the problem was first overcome by

replacing one of the waves by a strong helical DC magnetic field, and by replacing the stationary electrons by a high energy beam of electrons<sup>6</sup>. In the highly relativistic electron frame, the helical magnetic field looks like a coherent electromagnetic wave of very large amplitude. Using this idea, a group at Stanford University built a free electron amplifier<sup>10</sup>, and subsequently a free electron laser<sup>11</sup>, both working on the principle of stimulated Thomson scattering. The experimental results agree favorably with the quantum mechanical theories of the effect by Madey<sup>6</sup> and Colson<sup>9</sup>.

The announcement that the effect discussed by Kapitza and Dirac in 1933 had at last been observed caused a great stir. Of particular interest to many investigators was a remark that appeared in a paper by Madey, Schwettman and Fairbank<sup>12</sup>. They pointed out that in the low energy photon limit,  $\hbar\omega \ll m_e c^2$ , the quantum mechanical formula for the gain of a wave due to stimulated Thomson scattering did not contain Planck's constant, and hence might have a classical interpretation. A few years later, Hopf, Meystre, Scully, and Louisell<sup>13</sup> published the first paper using classical physics to obtain the correct quantum mechanical gain formula obtained by Colson<sup>9</sup>. Classically, the stimulated scattering is produced by a

bunching of the radiating particles along the direction of propagation of the amplified wave. As seen from the electron frame, this bunching is produced by the non-linear ponderomotive force of the two electromagnetic waves on the electrons. It causes the transverse current driven by the two waves to be non-uniform along the propagation direction of the amplified wave. This non-uniformity produces the equivalent of an array of phased antennas; by proper choice of the frequency difference between the two waves (as observed in the electron frame), this phased array can be made to take energy from one wave, and to put it into the other. The paper of Hopf et al.<sup>13</sup>, was accompanied by a flood of papers extending the classical results, or obtaining them in different ways<sup>14-22</sup>.

At the same time, there was great interest in using devices with periodic magnetic fields and intense high energy electron beams, (several of which pre-dated the Stanford device), to produce stimulated radiation by other mechanisms, like stimulated Raman scattering, stimulated Brillouin scattering, and cyclotron resonance<sup>23-38</sup>. In these processes, the two counterpropagating electromagnetic waves are not coupled through Thomson scattering by free electrons, but through coupling to a normal mode of the electron fluid or

mechanism is shown to be analogous to non-linear Landau damping. In Chapter 5, the results are summarized. In Appendix 1, the free electron amplifier experiment is discussed, and formulas are given for transforming quantities in the highly relativistic electron frame to the laboratory frame for comparing gain coefficients. In Appendix 2, formulas for the gain and dispersion functions discussed in Chapter 3 are given. In Appendix 3, a proposed experiment to observe stimulated Thomson scattering is described. Finally, in Appendix 4 are listed useful properties of the Jacobian Elliptic Functions used in Chapter 4.

## CHAPTER 2

## QUANTUM MECHANICAL THEORY OF STIMULATED THOMSON SCATTERING

In this chapter is presented a quantum mechanical theory of stimulated Thomson scattering following the approach of Madey<sup>6</sup>. Much of the theory presented here is due to L. W. Anderson and J. E. Lawler.

Consider two oppositely directed electromagnetic waves of frequencies  $\omega_0$  and  $\omega$ , and of matching polarization. The wave at frequency  $\omega_0$  will be called the pump wave, that at frequency  $\omega$  the probe wave. Each of the two waves is assumed to be very nearly monochromatic with narrow bandwidths  $\Delta\omega_0$  for the pump and  $\Delta\omega$  for the probe. At time  $t = 0$  they are suddenly turned on everywhere in a plasma of electron density  $n_e$ . The interaction of the waves with the electrons is allowed to last for a time  $\tau$  after which it is terminated.

This rather artificial situation has for its chief virtue simplicity. The more natural physical situation would be to allow two pulses of electromagnetic radiation to pass through each other; in that case different electrons in the plasma would interact with the two waves for different times, depending on where the electrons

were when the two pulses began to overlap. This more complicated situation can be analyzed in terms of the simpler situation considered here if each pulse contains many wavelengths of the radiation, for then when the two pulses overlap in some region, locally it is as if the interaction were simultaneously turned on over many wavelengths. The simple case may then be applied locally, using the interaction time appropriate for that location, to obtain the effect on the pulses.

The stimulated scattering of photons from the pump wave to the probe wave will now be studied. The total rate of Thomson scattering into a mode, including both spontaneous and stimulated photons, is  $\gamma'(n+1)$  where  $\gamma'$  is the rate of spontaneous Thomson scattering into the mode, and where  $n$  is the number of photons in the mode. The primary mechanism by which photons are lost from the mode is taken to be the inverse process in which the pump wave at  $\omega_0$  stimulates the scattering of photons out of the probe wave at  $\omega$ . Other loss mechanisms are ignored. The spontaneous rate per unit volume and per unit solid angle at which photons are Thomson backscattered from the pump wave is given by the equation

$$R = \frac{n_e r_e^2 I_0}{\hbar \omega_0} \quad (1)$$

where  $r_e$  is the classical electron radius. In Eq. (1) use has been made of the fact that the differential Thomson scattering cross section in the backward direction is  $r_e^2$ . The backscattered frequency is  $\omega' = \omega_0 - 2\hbar\omega_0/mc^2$  or, approximately,  $\omega' \approx \omega_0 - 2\hbar\omega_0^2/mc^2$ . Since the stimulated scattering lasts only for a time  $\tau$ , the scattered photons will have a band width  $\Delta\omega' \sim 1/\tau$  about  $\omega'$ . If  $\tau \gg 1/\Delta\omega_0$ , then the spectrum of the scattered radiation is determined solely by the interruption of the process after a time  $\tau$ . Call the normalized lineshape function for the scattered radiation  $g$ . The quantity  $Rg$  is the scattering rate into all modes at the frequency  $\omega'$  per unit volume per unit solid angle per unit frequency interval. Dividing  $Rg$  by  $\omega'^2/(2\pi c)^3$ , the number of modes per unit volume per unit solid angle per unit frequency interval will give  $\gamma'$ , the rate of Thomson scattering from the pump wave into a mode. Hence, the scattering rate into a mode of the probe wave is given by the expression

$$\begin{aligned}
 \gamma' &= \frac{(2\pi c)^3 R}{\omega^2} g \\
 &= \frac{8\pi^3 c^3 n_e r_e^2 I_o}{\hbar \omega^3} g(\omega - (\omega_o - 2\hbar\omega_o^2/mc^2))
 \end{aligned} \tag{2}$$

The net gain coefficient is gotten by subtracting the rate at which the pump wave stimulates scattering out of the probe wave. Replacing  $\omega_o$  and  $\omega$  by  $\bar{\omega} = (\omega + \omega_o)/2$  except when their difference is to be taken, and using  $\hbar\omega_o \ll mc^2$  yields for the net gain coefficient

$$\begin{aligned}
 \gamma\tau &= \frac{8\pi^3 c^3 n_e r_e^2 I_o \tau}{\hbar \bar{\omega}^3} [g(\omega - \omega_o + 2\hbar\omega_o^2/mc^2) - g(\omega - \omega_o - 2\hbar\omega_o^2/mc^2)] \\
 &\approx \frac{32\pi^3 n_e r_e^2 I_o c \tau}{m \bar{\omega}} \frac{d}{d\omega} g(\omega - \omega_o)
 \end{aligned} \tag{3}$$

The quantity  $I\gamma\tau$  is the total change in intensity of the probe wave due to the stimulated scattering.

In the homogeneously broadened limit, where the electron gas is very cold and the spectral broadening of the Thomson scattering is the result of the finite interaction time of the electrons with the two waves the normalized lineshape,  $g$ , is given by the equation<sup>7,9</sup>



$$g(\omega - \omega_0) = \frac{\tau}{2\pi} \frac{\sin^2[(\omega - \omega_0)\tau/2]}{[(\omega - \omega_0)\tau/2]^2} \quad (4)$$

and the net gain is

$$\gamma\tau = \frac{8\pi^3 n_e r_e^2 I_0 c \tau^3}{m \bar{\omega}} \frac{d}{d\eta} \left( \frac{\sin^2 \eta}{\eta^2} \right) \quad (5)$$

where  $\eta = (\omega - \omega_0)\tau/2$ .

In an inhomogeneously broadened system, where the spectral broadening of the Thomson scattering is the result of Doppler shifts from the distribution of velocities in the electron gas, and where  $f(v_z)$  is a normalized distribution function for electron velocities parallel to the light beam, the net gain is given by the equation

$$\begin{aligned} \gamma\tau &= \frac{32\pi^3 n_e r_e^2 I_0 c \tau}{m \bar{\omega}} \int_{-\infty}^{\infty} \frac{d}{d\omega} g(\omega - \omega_0 + 2\bar{\omega}v_z/c) f(v_z) dv_z \\ &= - \frac{16\pi^3 n_e r_e^2 I_0 c^2 \tau}{m \bar{\omega}^2} \int_{-\infty}^{\infty} dv_z g(\omega - \omega_0 + 2\bar{\omega}v_z/c) \frac{d}{dv_z} f(v_z) \quad (6) \end{aligned}$$

In the limit that inhomogeneous broadening dominates over homogeneous broadening, the net gain at, or near, line center is given by the equation

$$\gamma\tau = - \frac{8\pi^3 n_e r_e^2 I_0 c^3 \tau}{m \bar{\omega}^3} \left. \frac{d}{dv_z} f(v_z) \right|_{v_z = \frac{(\omega_0 - \omega)c}{2\omega_0}} \quad (7)$$

For a Maxwell-Boltzmann distribution function, the net gain is given by the equation

$$\gamma\tau = \frac{8\pi^{\frac{5}{2}} n_e r_e^2 I_0 c^3 \tau}{\bar{\omega}^3 k_B T} y e^{-y^2} \quad (8)$$

where  $y = \left( \frac{mc^2}{8k_B T} \right)^{\frac{1}{2}} \frac{\omega_0 - \omega}{\bar{\omega}}$ , (9)

T is the temperature of the electron gas, and  $k_B$  is Boltzmann's constant. Dreicer has discussed stimulated Thomson scattering<sup>4</sup>. His results can be used to derive Eq. (8) except for a difference of a factor of  $2\pi$ .

The gain mechanism of the free electron amplifier is essentially stimulated Thomson scattering<sup>10</sup>. In the free electron amplifier, a relativistic electron beam is incident on a transverse magnetic field. The field is

produced by currents in a double helix wound on a long pipe so that electrons shot down the pipe see a magnetic field whose direction rotates. When transformed to a frame of reference moving with the electrons, the spatially periodic magnetic field appears as a plane electromagnetic wave; it plays the role of the pump wave. In Appendix 1, it is shown that Eq. (5), when transformed to the highly relativistic electron frame, is the same as the gain formula for the free electron amplifier calculated by others<sup>9,13,19,21</sup>.

Since Planck's constant does not appear in the gain coefficient of the free electron laser, a classical approach might be expected to give the same result. This was first suggested by Madey et al<sup>12</sup>. Since that suggestion, many papers have appeared exploring the classical interpretation of both the free electron laser and stimulated Thomson scattering<sup>8,15,20,30,42,54,60-62</sup>. In the remainder of this thesis, a classical calculation is presented that yields the gain and dispersion of a plasma in the presence of two counter-propagating electromagnetic waves. The gain formula so obtained agrees, in the unsaturated limit, with that obtained quantum mechanically in this chapter. In addition, the classical calculation shows the relationship between stimulated Thomson scattering and stimulated Raman

scattering, and how the effect saturates by non-linear Landau damping.

## CHAPTER 3

## FLUID THEORY OF STIMULATED THOMSON SCATTERING

## A. Introduction

In this chapter a fluid calculation will be described that yields the same formula for the gain coefficient as that obtained quantum mechanically in Chapter 2. In addition, the close relationship between stimulated Thomson scattering and stimulated Raman scattering is clarified, and the effect of damping in the fluid equations on the gain coefficient is obtained. Besides the gain coefficient, the dispersion due to stimulated Thomson scattering is also obtained.

In spite of being able to take the limit of stimulated Raman scattering, namely  $\omega_0 - \omega = \omega_p$ , the results obtained here are difficult to compare with the rather large literature on this subject ( see for example Refs. 44 and 51 ). One of the reasons for this difficulty is that the usual approach is to Fourier transform the equations in space and time, whereas in this chapter an initial value problem is solved. This procedure produces transient solutions, and it is these transients that are of interest in connection with

stimulated Thomson scattering. A second reason is that in Raman scattering an incident electromagnetic wave decays into a plasma wave and a scattered electromagnetic wave. Here, two externally produced electromagnetic waves are incident on a plasma and act to induce a plasma wave if the resonance condition is satisfied. Thus, the approach taken here is more in the spirit of optical mixing of electromagnetic waves or of beat heating in plasmas<sup>30,33,35,37,39,41,43,50,51,64</sup>.

#### B. A Perturbation Expansion of the Electron Fluid Equations and Maxwell's Equations

It is assumed that two counterpropagating electromagnetic waves of the same polarization are incident on a neutralized electron fluid. At time  $t=0$  the two waves are instantaneously turned on, everywhere in space, in the previously undisturbed fluid. Effects due to the sudden turning on are ignored. The development in time of the intensities and relative phases of the two waves are obtained by solving the electron fluid equations and Maxwell's equations. It is assumed that the timescale for the amplitudes and phases of the waves to change is much longer than the period of either of the two waves, and in fact is much longer than

$1/\delta\omega$ , where  $\delta\omega = \omega_0 - \omega$ . The vector potential describing the two waves is written in the form

$$\vec{A}_T = \left[ \frac{c}{\omega_0} E_0 (1 + \theta_0(t)) \sin(k_0 z - \omega_0 t + \phi_0(t)) + \frac{c}{\omega} E (1 + \theta(t)) \sin(kz + \omega t + \phi(t)) \right] \hat{x} \quad (10)$$

where  $\theta$  and  $\theta_0$  represent the change in the wave amplitudes and where  $\phi$  and  $\phi_0$  represent the change in the phases of the waves. Note that  $\vec{A}_T$  satisfies the Coulomb gauge condition,  $\vec{\nabla} \cdot \vec{A}_T = 0$ . At time  $t=0$ ,  $\theta$ ,  $\phi$ ,  $\theta_0$ , and  $\phi_0$  are all zero. These quantities are assumed to satisfy the following ordering.

$$\dot{\theta}, \dot{\phi}, \dot{\theta}_0, \dot{\phi}_0 \ll \delta\omega \quad (11)$$

The electron fluid is taken to be described by the equation of motion,

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\frac{e}{m} \left( \vec{E} + \frac{\vec{v}}{c} \times \vec{B} \right) - \frac{1}{m} \nabla p \quad (12)$$

the equation of continuity,

$$\frac{\partial n}{\partial t} + \vec{\nabla} \cdot (n \vec{v}) = 0 \quad (13)$$

an equation of state,

$$\left( \frac{p}{p_0} \right) = \left( \frac{n}{n_0} \right)^{\gamma^*} \quad (14)$$

and Maxwell's equations,

$$\vec{\nabla} \cdot \vec{E} = -4\pi e (n - n_0) \quad (15)$$

$$\vec{\nabla} \cdot \vec{B} = 0 \quad (16)$$

$$\vec{\nabla} \times \vec{B} = -\frac{4\pi}{c} e n \vec{v} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \quad (17)$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t} \quad (18)$$

The quantity  $\vec{v}$  is the fluid velocity,  $p$  is the pressure,  $n$  is the electron fluid density,  $m$  is the electron mass, and  $\vec{E}$  and  $\vec{B}$  are the electric and magnetic fields. The variable  $\gamma^*$  represents the adiabatic exponent and  $n_0$  is the density of the neutralizing positive charge. There is a wealth of information contained in these equations. For the purposes of this calculation, it is sufficient to make a perturbation expansion of the equations. To lowest order, the fluid is taken to respond only to the electric fields of the two external electromagnetic waves; in the next order, the motion of the fluid along



the z-direction, the direction of propagation of the waves, is considered. Hence, the following perturbation scheme is adopted.

$$\vec{v} = \vec{v}_0 + \delta v \hat{z} \quad (19)$$

$$n = n_0 + \delta n \quad (20)$$

$$\vec{E} = \vec{E}_T + \delta E \hat{z} \quad (21)$$

$$\vec{B} = \vec{B}_T \quad (22)$$

The quantity  $\vec{v}_0$  is the fluid velocity driven by the external electric fields while  $\delta v$ ,  $\delta n$ , and  $\delta E$  describe the motion of the fluid in the z-direction and the longitudinal electric field produced by that motion.

Equations (19)-(22) are substituted into the fluid equations and Maxwell's equations, Eqs.(15)-(18), and the system of equations is linearized to obtain the set of equations

$$\frac{\partial \vec{v}_0}{\partial t} = -\frac{e}{m} \vec{E}_T \quad (23)$$

$$\frac{\partial \delta v}{\partial t} = -\frac{e}{mc} (\vec{v}_0 \times \vec{B}_T)_z - \frac{e}{m} \delta E - \frac{\gamma^* k_B T}{m n_0} \frac{\partial \delta n}{\partial z} \quad (24)$$

$$\frac{\partial \delta E}{\partial z} = -4\pi e \delta n \quad (25)$$

$$\frac{\partial^2 \vec{A}_T}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2 \vec{A}_T}{\partial t^2} - \frac{\omega_p^2}{\omega^2} \vec{A}_T = \frac{4\pi}{c} e \delta n \vec{v}_0 \quad (26)$$

$$(27)$$

This procedure is similar to that followed in Refs. 38 and 64. In obtaining these equations it was assumed that the fluid and the fields are homogeneous in the x- and y-directions so that all derivatives with respect to x and y vanish.

### C. The Motion of the Electron Fluid Driven by the Ponderomotive Force for Weak Plasma Response

With the ordering of Eq. (11), this set of equations may be solved by iteration: find the perturbed fluid quantities assuming that  $\theta$ ,  $\theta_0$ ,  $\phi$ , and  $\phi_0$  are negligible,

then substitute the perturbed fluid quantities into the wave equation, Eq. (27), to determine  $\theta$ ,  $\theta_0$ ,  $\phi$ , and  $\phi_0$ . Equation (23) is solved first to obtain

$$\vec{v}_0 = \frac{e}{mc} ( \vec{A}_T - \vec{A}_T|_{t=0} ) \quad (28)$$

Note that the requirement that the fluid be initially undisturbed has been applied. This expression for  $\vec{v}_0$  is substituted into the right hand side of Eq. (24). The term containing  $\vec{v}_0$  in Eq. (24) acts as an external driving force in the longitudinal equation of motion, and is simply the familiar ponderomotive force together with a term due to the initial conditions:

$$-\frac{e}{mc} \vec{v}_0 \times \vec{B}_T = -\frac{e^2}{2m^2c^2} \vec{\nabla} A_T^2 + \frac{e^2}{m^2c^2} \vec{A}_T|_{t=0} \times (\vec{\nabla} \times \vec{A}_T) \quad (29)$$

The ponderomotive force is the force that produces stimulated Thomson scattering, and must be treated with care. The importance of the ponderomotive force may be seen by expanding  $A_T^2$  in the following way.

$$\begin{aligned}
A_T^2 &= E_0^2 \frac{c^2}{\omega_0^2} \sin^2(k_0 z - \omega_0 t) + E^2 \frac{c^2}{\omega^2} \sin^2(kz + \omega t) \\
&+ \frac{E_0 E c^2}{\omega \omega_0} [ \cos((k_0 - k)z - (\omega_0 + \omega)t) - \cos((k_0 + k)z - \delta \omega t) ] \quad (30)
\end{aligned}$$

The first three terms in this expression are high phase velocity terms and can have little effect on the electron motion, but the fourth term has phase velocity much less than the speed of light; it can influence the electron fluid more strongly than the other terms. Assuming that the high phase velocity terms in Eq. (30) as well as similar terms arising from the initial value term in Eq. (29) are negligible, the ponderomotive force term in the fluid equation of motion may be written in the form

$$\frac{\vec{F}_P}{m} = -\frac{e}{mc} (\vec{v}_0 \times \vec{B}_T) \approx -\frac{1}{2} \frac{\Omega^2 c}{\bar{\omega}} \sin((k_0 + k)z - \delta \omega t) \hat{z} \quad (31)$$

$$\text{where } \Omega^2 = \frac{(k_0 + k) e^2 E E_0}{m^2 c \bar{\omega}} \approx \frac{2 E_0 E e^2}{m^2 c^2}$$

and where  $\bar{\omega} = (\omega + \omega_0)/2$ . Substituting this term into Eq. (24) and Fourier transforming Eqs. (24)-(26) in  $z$  with respect to the Fourier transform variable  $k'$  yields the following set of equations.

$$\frac{\partial}{\partial t} \delta \tilde{v} = \frac{\tilde{F}_p}{m} - \frac{e}{m} \delta E - \frac{i k' \gamma^* k_B T}{m n_0} \delta n \quad (32)$$

$$\frac{\partial}{\partial t} \delta n + i k' n_0 \delta \tilde{v} = 0 \quad (33)$$

$$i k' \delta E = -4\pi e \delta n \quad (34)$$

Combining these equations yields the single equation for  $\delta n$ ,

$$\frac{d^2}{dt^2} \delta n + \left( \omega_p^2 + \frac{\gamma^* k_B T}{m} k'^2 \right) \delta n = - \frac{i k' n_0}{m} \tilde{F}_p \quad (35)$$

The left hand side by itself would simply yield the familiar Bohm-Gross dispersion relation for plasma waves. The right hand side represents the effect of the low frequency part of the non-linear ponderomotive force. Mathematically, the equation is simply that describing the driven harmonic oscillator.

Equation (35) does not include any damping effects. Since plasma waves may be damped either collisionally or

by Landau damping, a phenomenological damping term is added to Eq. (35) to obtain the equation

$$\frac{d^2}{dt^2} \tilde{\delta n} + \Gamma \frac{d}{dt} \tilde{\delta n} + \omega'^2 \tilde{\delta n} = \frac{k' n_o \Omega_c^2}{4 \bar{\omega}} [ \delta(k' - k_o - k) e^{-i\delta\omega t} - \delta(k' + k_o + k) e^{i\delta\omega t} ] \quad (36)$$

$$\text{where } \omega'^2 = \omega_p^2 + \frac{\gamma^* k_B T}{m} (k_o + k)^2$$

Equation (36) is simply solved, but working with its solutions is very messy. However, the technique used to find the time dependent electromagnetic field amplitudes and phases can be illustrated in a very simple case which is interesting in its own right. The more complicated results obtained by working with Eq. (36) are given in Appendix 2 and in the figures. The simple case is to assume that  $\Gamma$  and  $\omega'$  are both zero. In this case Eq. (36) can be immediately integrated to obtain

$$\tilde{\delta n} = - \frac{k' n_o^2 c}{4 \delta \omega^2 \bar{\omega}} [ \delta(k' - k_o - k) e^{-i\delta\omega t} - \delta(k' + k_o + k) e^{i\delta\omega t} ] + Ct + D \quad (37)$$

where C and D are constants. That the fluid be initially undisturbed imposes the conditions

$$\delta \tilde{n} \Big|_{t=0} = 0 \quad (38)$$

$$\frac{d}{dt} \delta \tilde{n} \Big|_{t=0} = 0 \quad (39)$$

which determine C and D. Equation (37) is now inverse-Fourier transformed in  $k'$  to obtain the desired solution.

$$\delta n = - \frac{n_o \Omega^2}{\delta \omega^2} [ \cos( (k_o+k)z - \delta \omega t ) - \delta \omega t \sin(k_o+k)z - \cos(k_o+k)z ] \quad (40)$$

#### D. The Gain and Dispersion of the Waves

Having solved for the fluid motion, the electromagnetic fields may now be determined. Substituting Eq. (10) into the wave equation, Eq. (27), and applying the ordering of Eq. (11) gives for Eq. (27)

$$\begin{aligned}
& \left[ \left( -\frac{k_0^2 c}{\omega_0} + \frac{\omega_0}{c} - \frac{\omega_0^2}{c\omega_0} \right) - \frac{2\dot{\phi}_0}{c} \right] E_0 \sin(k_0 z - \omega_0 t + \phi_0) \\
& + \frac{2E_0}{c} \dot{\theta}_0 \cos(k_0 z - \omega_0 t + \phi_0) \\
& \hspace{20em} (41) \\
& + \left[ \left( -\frac{k^2 c}{\omega} + \frac{\omega}{c} - \frac{\omega^2}{c\omega} \right) + \frac{2\dot{\phi}}{c} \right] E \sin(kz + \omega t + \phi) \\
& - \frac{2E}{c} \dot{\theta} \cos(kz + \omega t + \phi) = \frac{4\pi}{c} e \delta n v_{ox}
\end{aligned}$$

Expressions for  $\theta$ ,  $\phi$ ,  $\theta_0$ , and  $\phi_0$  may be obtained by multiplying Eq. (41) by an appropriate factor and integrating over time and space. The time integration is assumed to be over a time,  $t$ , long compared to the wave period of either wave, but short compared to the time  $1/\delta\omega$ . The integration in  $z$  is taken to be over a region long compared with the length  $1/(k-k_0)$ . The procedure is illustrated by multiplying Eq. (41) by  $\cos(kz) \cos(\omega t)$  and integrating as described above. Equation (41) becomes

$$-\frac{E}{2c} \dot{\theta} = \frac{4\pi}{c} e \frac{1}{Lt} \int_{-L/2}^{L/2} dz \int_0^t dt' \delta n v_{ox} \cos kz \cos \omega t' \quad (42)$$

Multiplying by  $\cos(kz)\sin(\omega t)$  and integrating yields



$$\frac{1}{4} \left( -\frac{k^2 c^2}{\omega} + \frac{\omega}{c} - \frac{\omega_p^2}{c\omega} \right) E + \frac{E}{2c} \dot{\phi} =$$

$$\frac{4\pi}{c} e \frac{1}{L} \int_{-L/2}^{L/2} dz \int_0^t dt' \delta n v_{ox} \cos kz \sin \omega t' \quad (43)$$

Carrying out the operations on the right hand sides of Eqs. (42) and (43), and equating terms of like order, yields the familiar dispersion relation for electromagnetic waves in a plasma and differential equations for  $\theta$  and  $\phi$ .

$$\omega^2 = \omega_p^2 + k^2 c^2 \quad (44)$$

$$\dot{\theta} = \frac{4\pi^2 n_o r_e^2 I_o c}{m \omega} P(t) \quad (45)$$

$$\dot{\phi} = -\frac{4\pi^2 n_o r_e^2 I_o c}{m \omega} Q(t) \quad (46)$$

where

$$P = \frac{4}{\delta\omega^2} ( \sin\delta\omega t - \delta\omega t \cos\delta\omega t ) \quad (47)$$

$$Q = \frac{4}{\delta\omega^2} ( \cos\delta\omega t - 1 + \delta\omega t \sin\delta\omega t ) \quad (48)$$

$$I_0 = \frac{c}{8\pi} E_0^2 \quad (49)$$

The differential equations are easily integrated over the time of interaction,  $\tau$ , to obtain

$$\theta = \frac{4\pi^2 n_0 r_e^2 I_0 c}{m \bar{\omega}} J \quad (50)$$

$$\phi = - \frac{4\pi^2 n_0 r_e^2 I_0 c}{m \bar{\omega}} K \quad (51)$$

where

$$J = \tau^3 \frac{d}{d\eta} \left( \frac{\sin^2 \eta}{\eta} \right) \quad (52)$$

$$K = \tau^3 \frac{d}{d\eta} \left( \frac{1}{\eta} - \frac{\sin 2\eta}{2\eta^2} \right) \quad (53)$$

and where  $\eta = -\delta\omega\tau/2$ .

The functions  $J$  and  $K$  given above are displayed in Figures 1 and 2.

A similar calculation shows that

Figure 1

The Gain Curve for Unsaturated  
Stimulated Thomson Scattering

The function  $-J(\eta) = -\tau^3 \frac{d}{d\eta} \left( \frac{\sin^2 \eta}{\eta^2} \right)$  is displayed for  
 $\tau = 1$ .

FIGURE 1

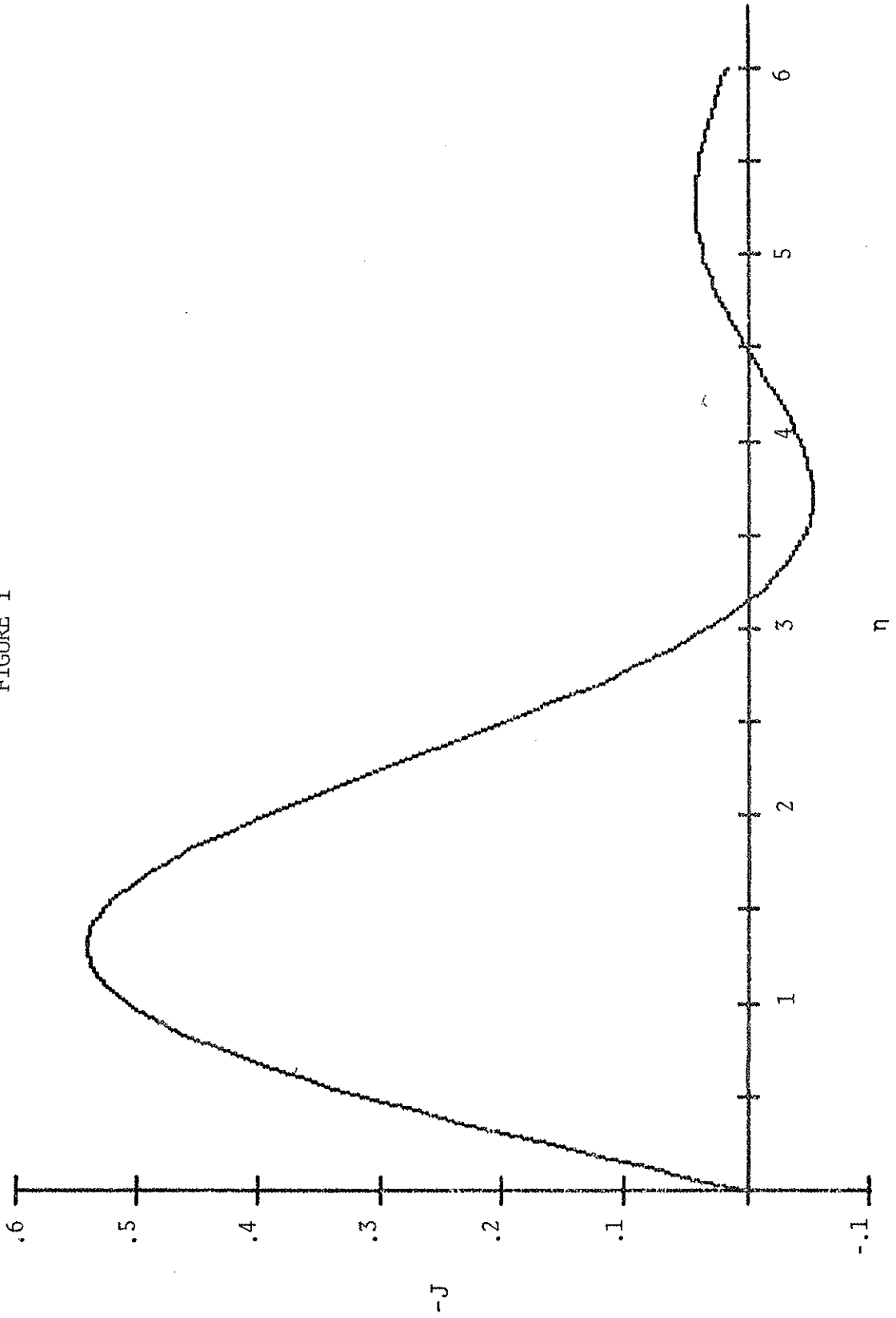
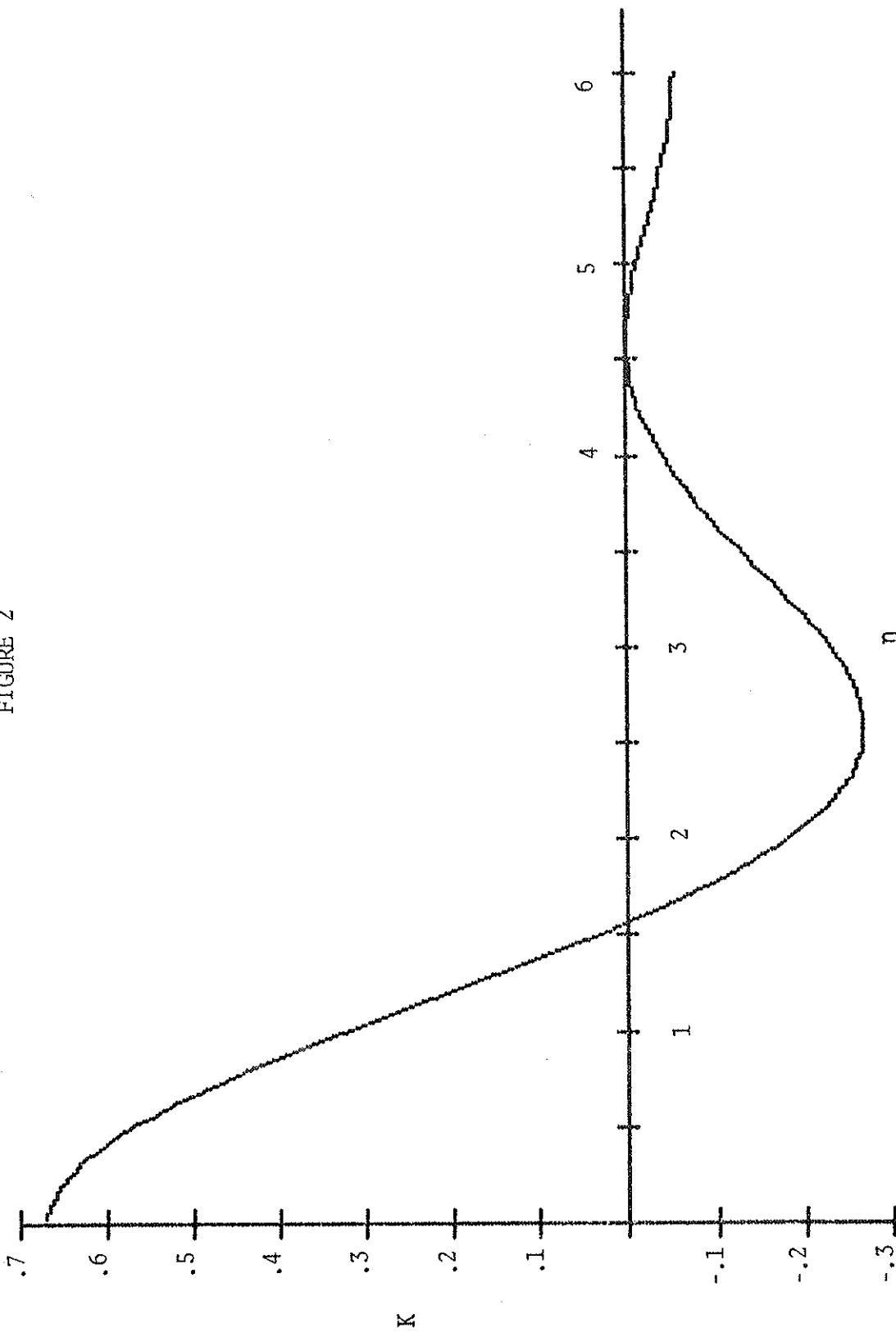


Figure 2

The Dispersion Curve for Unsaturated  
Stimulated Thomson Scattering

The function  $K(\eta) = \tau^3 \frac{d}{d\eta} \left( \frac{1}{\eta} - \frac{\sin 2\eta}{2\eta^2} \right)$  is displayed for  
 $\tau = 1$ .

FIGURE 2



$$E_0^2 \theta_0 = - E^2 \theta \quad (54)$$

$$E_0^2 \phi_0 = - E^2 \phi \quad (55)$$

Equation (54) shows that energy is conserved between the two waves; to the order of approximation taken here, the fluid acts only to transfer energy from one wave to the other. This is not strictly true, of course, for the fluid obviously gains energy in the process described above. The amount of energy gained by the fluid is of order  $\delta\omega/\bar{\omega}$  compared with the energy transferred between the two waves. Terms of this order have been consistently neglected in this calculation.

#### E. Comparison with the Quantum Mechanical Result;

##### The Kramers-Kronig Relations

To make the connection between Eq. (50) and the quantum mechanical result, note that  $\Upsilon\tau$  given in Eq. (5) represents the fractional gain in intensity due to stimulated Thomson scattering acting for a time  $\tau$ . Since  $I \propto E^2(1 + 2\theta)$ ,

$$\gamma\tau = 2\theta = \frac{8\pi^2 n_o r_e^2 I_o c \tau^3}{m \bar{\omega}} \frac{d}{d\eta} \left( \frac{\sin^2 \eta}{\eta^2} \right) \quad (56)$$

in agreement with the quantum mechanical result.

The time derivatives of  $\theta$  and  $\phi$  may be given an interpretation as real and imaginary parts of a non-linear index of refraction. Since  $\dot{\theta}$  and  $\dot{\phi}$  are small and vary slowly in time compared to  $\omega t$ , and since  $(1 + \theta) \approx \exp(\theta) = \exp(\int_0^T \dot{\theta} dt)$ , the vector potential of the probe wave can be expressed as follows.

$$\vec{A} = \frac{c}{\omega} \vec{E} e^{i(kz + \int_0^T \omega^* dt)} \quad (57)$$

where  $\omega^* = \omega + \dot{\phi} - i\dot{\theta}$

The index of refraction may now be written in the form

$$n^2 = \frac{k^2 c^2}{\omega^{*2}} \approx 1 - \frac{\omega_p^2}{\omega^2} - \frac{2\dot{\phi}}{\omega} + \frac{2i\dot{\theta}}{\omega} \quad (58)$$

It is interesting to note that  $\dot{\theta}$  and  $\dot{\phi}$ , interpreted in this way, are connected by the Kramers-Kronig relations with respect to  $\omega$ . The Kramers-Kronig relations apply to linear, time independent systems, and hence might not be expected to apply to the situation



considered here. The question of linearity is answered by noting that although the interaction of the two waves is non-linear, it is non-linear in a special way. The interaction is characterized by the quantity  $\Omega^2 \propto EE_0$ ; this is a non-linear term, but is linear separately in the unperturbed amplitudes  $E$  and  $E_0$ . Another way of saying the same thing is to say that the medium seen by the probe wave consists of the electrons plus the pump wave. Viewed in this way, the interaction of the probe wave with the "medium" is linear. The problem that the medium is time dependent is solved by showing that the same results are obtained from a different, albeit contrived, medium which is time independent. Note that Equation (36) is linear in the density of the medium,  $n_0$ , and in the density perturbation  $\delta n$ . Suppose now that instead of one medium of density  $n_0$ , there are many such media of smaller densities in the presence of the two waves. At some time, one of the media has all of its density perturbations smoothed out. It is then allowed to respond for a time  $\tau$ , at which time it is once again made smooth and allowed to respond again for a time  $\tau$ . The same thing happens over and over for all of the media, but the smoothing times for the various media are randomly scattered over the time interval  $\tau$ . At the end of each time interval, the medium which is smoothed out

will have produced a change in the probe wave amplitude. From Eq. (57), the change can be represented, for small  $\phi$ , by

$$\Delta \vec{E} = \vec{E} e^{i(kz + \omega t)} (\theta + i\phi) \quad (59)$$

Thus for this system,  $\theta$  and  $\phi$  are simply the real and imaginary parts of the linear response function for the probe wave. They are to be thought of as functions of  $\omega$  with  $\omega_0$  and  $\tau$  as parameters. When viewed in this way, it is clear that  $\theta$  and  $\phi$  should be connected by the Kramers-Kronig relations. Since these relations are linear relations between analytic functions, and since  $\tau$  is simply a parameter,  $\dot{\theta}$  and  $\dot{\phi}$  should also be connected by the Kramers-Kronig relations.

#### F. The Gain and Dispersion for General Plasma Response

The effect of including nonzero  $\omega'$  and  $\Gamma$  will now be considered. Equation (36) is easy, if messy, to solve. The solution for  $\delta n$  is combined with  $\vec{v}_0$  to obtain the non-linear correction to the transverse current, and hence to obtain  $\theta$ ,  $\phi$ ,  $\theta_0$ , and  $\phi_0$  from the procedure outlined in Eqs. (42)-(53). Carrying out this program

yields more complicated expressions for J and K. These expressions, together with the solution of Eq. (36) for  $\delta n$  are given in Appendix 2.

Two cases will now be considered. In case (i) the interaction time  $\tau$  is held fixed, as are  $\Gamma$  and  $\omega'$ , while  $\delta\omega$  is allowed to vary. In case (ii), the difference frequency,  $\delta\omega$ ,  $\Gamma$ , and  $\omega'$  are held fixed while  $\tau$  is allowed to vary. Case (i) is obtained in arrangements like the free electron amplifier experiment where the difference frequency is varied by changing the electron beam energy while the other parameters are held fixed. In case (ii) the time development of the amplitudes and phases of the electromagnetic waves is considered, a natural thing to look at given the way the problem has been solved.

Figures 3-8 show the functions J and K for case (i) with  $\tau = 1$  and with  $0 \leq \delta\omega\tau \leq 12$ . Note that in all cases the maximum gain occurs for  $\omega'\tau \ll 1$  and  $\Gamma\tau \ll 1$ , i.e. for simple stimulated Thomson scattering. Note also that there is very little difference between the cases of critical damping and overdamping of the plasma wave. If  $\tau$  has a value other than  $\tau = 1$ , the following identity allows the desired value of J or K to be found in terms of the case  $\tau = 1$ .

Figure 3

The Gain Curve for Underdamping and  $\tau$  Fixed

The function  $J$  is displayed vs.  $\delta\omega\tau$  for  $\tau = 1$ , for  $\Gamma \ll \omega'$ , and for 3 values of  $\omega'$ :

————  $\Gamma = .0001 \quad \omega' = .001$   
———  $\Gamma = .001 \quad \omega' = 2$   
————  $\Gamma = .001 \quad \omega' = 4$

FIGURE 3

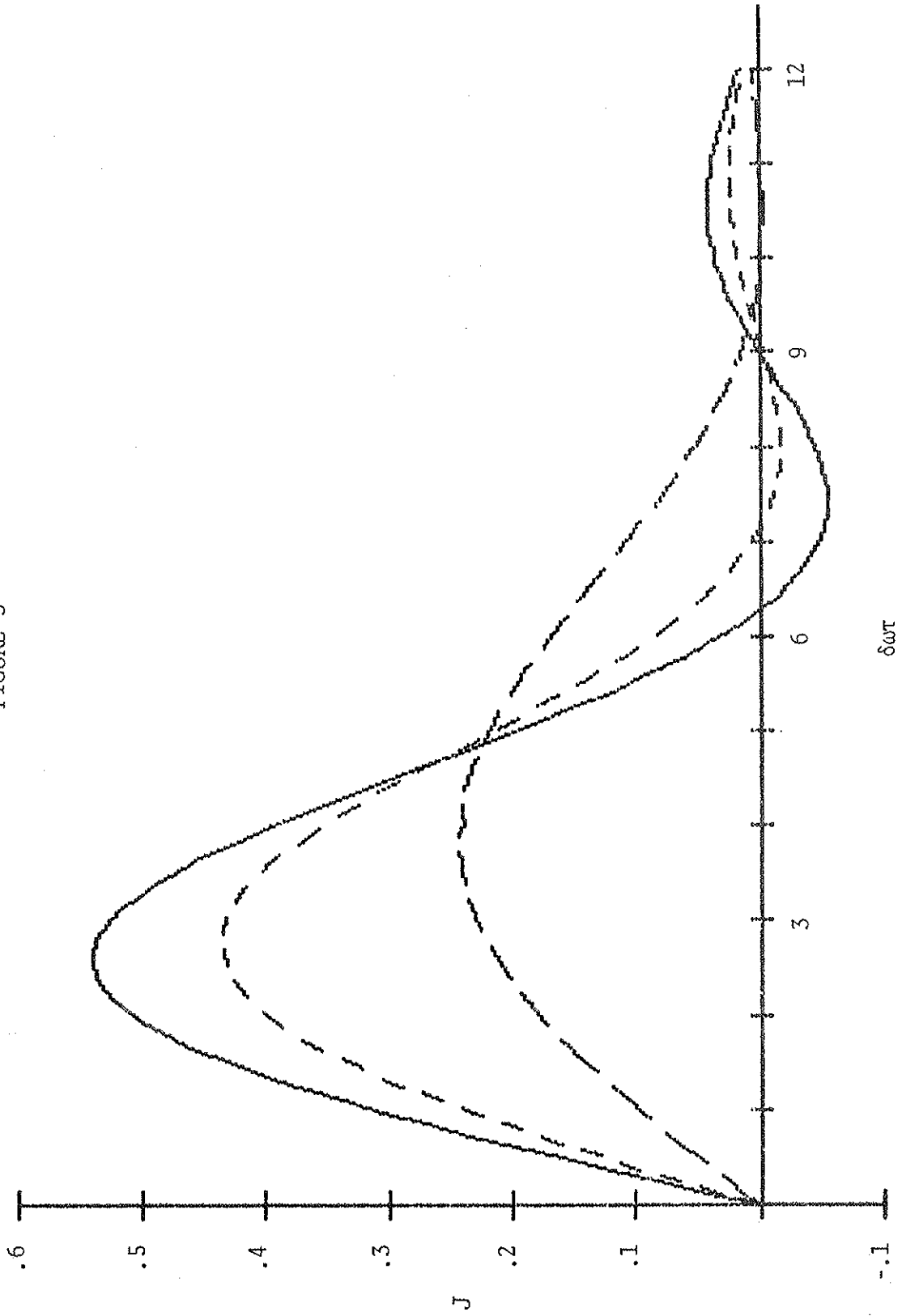


Figure 4

The Gain Curve for Critical Damping and  $\tau$  Fixed

The function  $J$  is displayed vs.  $\delta\omega\tau$  for  $\tau = 1$ , for  $\omega' = \frac{1}{2}\Gamma$ , and for 3 values of  $\Gamma$ :

————  $\Gamma = .019 \quad \omega' = .01$   
———  $\Gamma = 1.999 \quad \omega' = 1$   
————  $\Gamma = 3.999 \quad \omega' = 2$

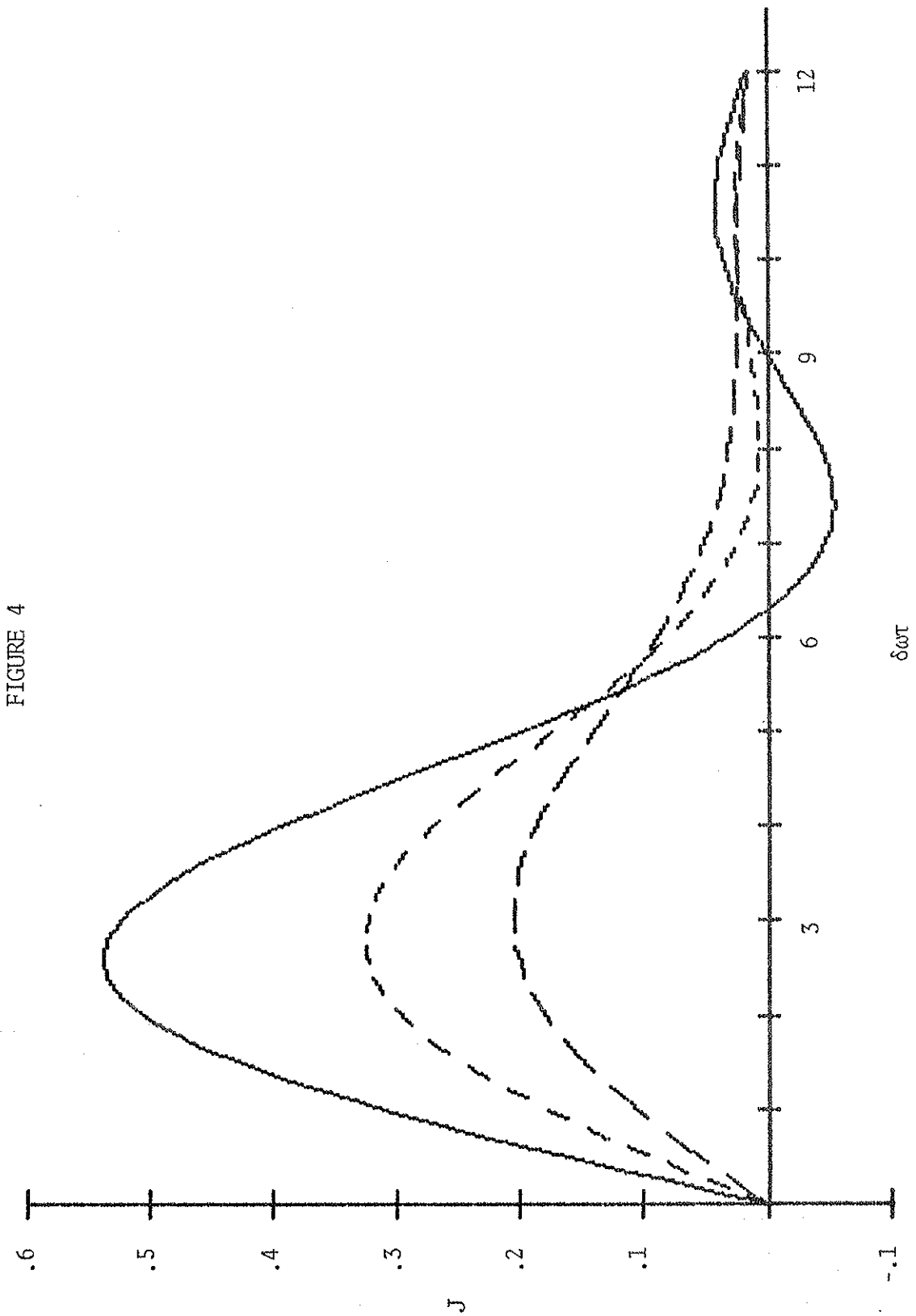


FIGURE 4

Figure 5

The Gain Curve for Overdamping and  $\tau$  Fixed

The function  $J$  is displayed vs.  $\delta\omega\tau$  for  $\tau = 1$ , for  $\omega' \ll \Gamma$ , and for 3 values of  $\Gamma$ :

—————	$\Gamma = .002$	$\omega' = .0001$
———	$\Gamma = 2$	$\omega' = .01$
—————	$\Gamma = 4$	$\omega' = .01$



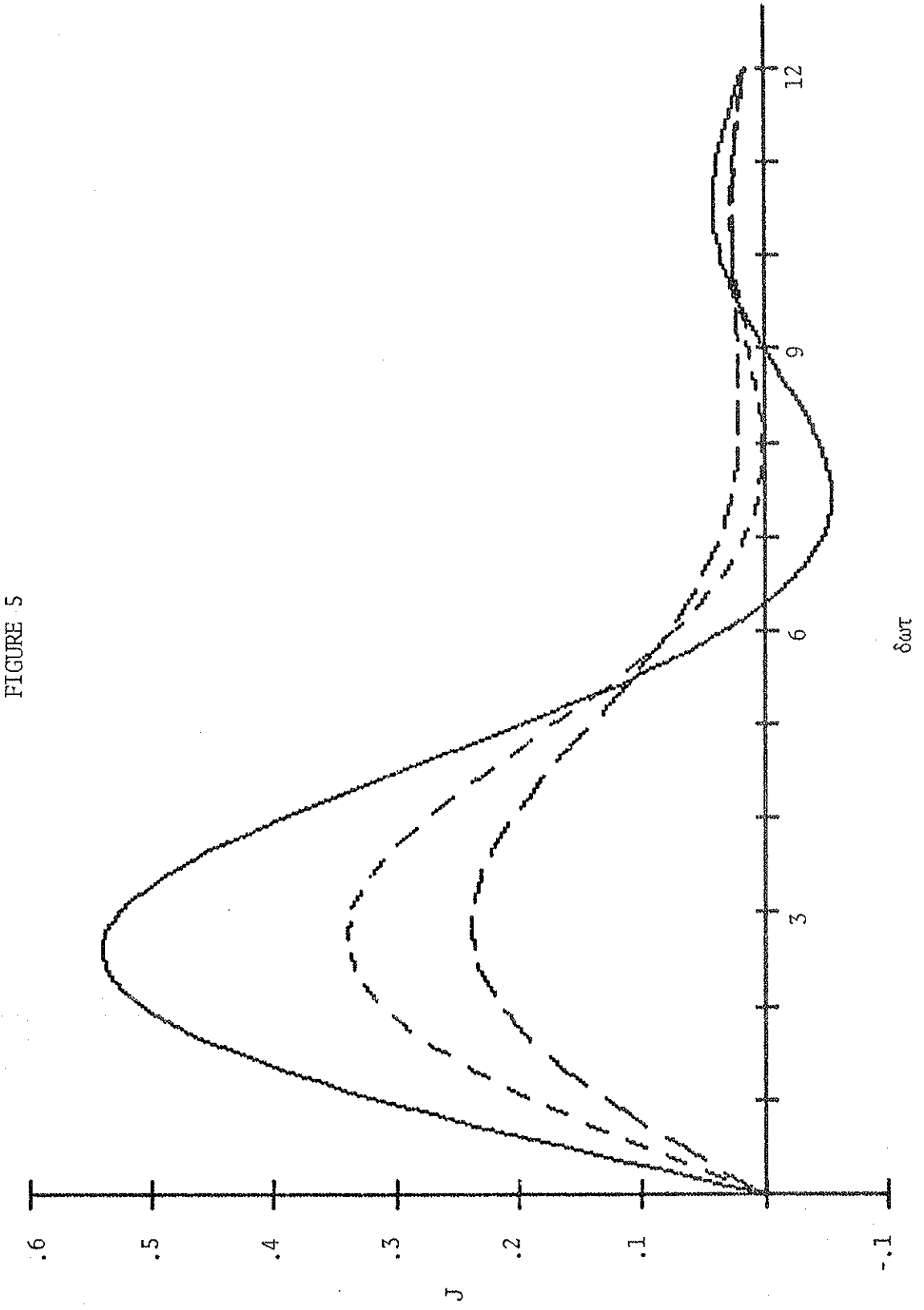


FIGURE 5

Figure 6

The Dispersion Curve for Underdamping and  $\tau$  Fixed

The function  $K$  is displayed vs.  $\delta\omega\tau$  for  $\tau = 1$ , for  $\Gamma \ll \omega'$ , and for 3 values of  $\omega'$ :

————  $\Gamma = .001 \quad \omega' = .01$

— — —  $\Gamma = .001 \quad \omega' = 2$

————  $\Gamma = .001 \quad \omega' = 4$